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Physics and mathematics of dispersion-managed optical solitons

Physique et mathématique des solitons optiques managés en dispersion

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Abstract

We review the main physical and mathematical properties of dispersion-managed (DM) optical solitons. Theory of DM solitons can be presented at two levels of accuracy: first, simple, but nevertheless, quantitative models based on ordinary differential equations governing evolution of the soliton width and phase parameter (the so-called chirp); and second, a comprehensive path-average theory that is capable of describing in detail both the fine structure of DM soliton form and its evolution along the fiber line. An analogy between DM soliton and a macroscopic nonlinear quantum oscillator model is also discussed. ***To cite this article: S.K. Turitsyn et al., C. R. Physique 4 (2003).***

Résumé

Nous passons en revue les principales propriétés physiques et mathématiques des solitons dits managés par la dispersion (DM). La théorie des solitions DM peut être développée selon deux niveaux de précision : le premier relève de modèles simples, mais toutefois quantitatifs, tels que basés sur des équations différentielles ordinaires gouvernant les deux paramètres solitons que sont la largeur temporelle et la phase (le soit-disant 'chirp' ou dérive temporelle de fréquence) ; le deuxième relève d'une théorie poussée de cheminement-moyen, laquelle est en mesure de décrire en détail et la structure fine de l'enveloppe du soliton DM, et son évolution tout au long de la ligne de fibre. Nous présentons également une discussion sur une analogie entre le DM soliton et un modèle d'oscillateur quantique non-linéaire à l'échelle macroscopique. ***Pour citer cet article : S.K. Turitsyn et al., C. R. Physique 4 (2003).***

Keywords: Optical soliton; Fiber communications; Dispersion-management

Mots-clés : Solitons optiques ; Communications à fibres optiques ; Management de dispersion

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1. Introduction

The soliton is one of the fundamental unifying ideas in modern theoretical physics and mathematics [1–10]. During the past few decades soliton theory has been applied to numerous practical and fundamental problems in areas as diverse as hydrodynamics, plasma, nonlinear optics, molecular biology, field theory, and astrophysics. Because of their stability and robustness, solitons provide a convenient and adequate language to describe many nonlinear phenomena. Note that the term soliton (formed from Latin *solitarius* – solitary) is used in a variety connotations in physics and mathematics. In the mathematical literature this term is typically associated with particle-like solutions of integrable equations that interact elastically and regain their forms after collisions. In physics and in applications we usually are not really concerned with the integrability (and hence with a strict definition of soliton) of the mathematical models involved, because anyway, a more careful consideration of practical perturbations or realistic boundary conditions leads to non-integrable systems. Fortunately, many characteristics of the soliton that are important for applications are not related to its strict mathematical definition and integrability. Non-integrable nonlinear models can possess solutions corresponding to spatially or temporally coherent, localized structures. If such localised objects/solutions are stable and robust, or at least are observable, having a reasonably large decay time (or distance), they are often also called solitons in the physical literature. Stable particle-like behavior is the key feature in such a physical definition of a soliton. We use here the term soliton for a localized (in time) electromagnetic wave that can propagate without significant distortion of its form even in the presence of a substantial nonlinear response of the waveguide medium.

An impressive practical implementation of the soliton concept has been achieved in fiber optics, where soliton pulses are used as the information carriers (elementary 'bits') to transmit digital signals over long distances. Optical soliton research, full of innovative spirit, has recently arrived at the stage of a first real-world implementation of the soliton concept in communication systems. Realization of soliton-based transmission has clearly demonstrated how the results of the fundamental soliton theory can be successfully exploited in very important practical applications. On the other hand, the experimental implementation of DM soliton fiber transmission lines and the increasing demand for capacities of communication systems has stimulated further research in soliton theory.

Linear optical fiber communication technologies are essentially based on the same principles as radio frequency systems. Soliton-based (or general *nonlinear*) fiber communication systems are fundamentally different, because they make positive use of such an inherent fiber property as nonlinearity. Rapid progress in nonlinear lightwave communications is stimulated by increasing demand for telecommunications services. Practical and research interest is directed mostly toward two main goals: development of effective high capacity long-haul transmission systems and the upgrade of existing terrestrial fiber networks. There are two principal approaches to overcome these limitations: in the first (that can be called 'linear') both the chromatic dispersion and nonlinearity are considered to be detrimental factors while in the second the nonlinear and dispersive effects are counterbalanced (such systems can be called 'nonlinear'). Nonlinear effects that are detrimental in the 'linear' systems can be used to improve transmission characteristics of optical communication systems. For instance, in soliton transmission the effects of nonlinearity and chromatic dispersion are balanced making positive use of the nonlinearity. Recall that there are three major factors that cause optical signal degradation and distortion in long-haul high bit-rates fiber communication systems: fiber loss, group-velocity dispersion (GVD) and nonlinearity. Signal power attenuation can be compensated using the optical fiber amplifiers (though recovery is not complete, because amplified spontaneous emission noise is added to the signal, degrading signal-to-noise ratio). Revolutionary development of the *nonlinear* lightwave communications has been triggered by advent and deployment of optical amplifiers [11] providing periodic amplification of optical signals. Until the advent of the erbium-doped fiber amplifier (EDFA) optical signals were regenerated electronically to overcome the attenuation in the silica fiber. Electronic regenerators have two important drawbacks: they are expensive and they limit system performance, because each regenerator can operate at only one predetermined bit-rate, in one data modulation format and at one operating wavelength. Because the EDFA amplifier has many important advantages (such as large bandwidth, high gain, simplicity and others) over optoelectronic regenerators, they quickly became the amplifier of choice in communication systems. As a result, fiber loss is no longer a major limitation in optical fiber transmission and the performance of optical amplifier systems is then limited by chromatic dispersion and nonlinearity. Note that whereas a regenerator re-creates a perfect digital output signal, the fiber amplifier uses whatever it receives. Therefore, dispersive pulse broadening and other degrading effects are accumulated along a fiber line.

2. Physical medium: fiber waveguide dispersion and nonlinearity

A light pulse is an electromagnetic wave packet built from a continuum of elementary optical carriers oscillating at different frequencies. In other words, any optical wave-packet contains a range of frequency components. Since any optical fiber is a dispersive medium, each of these spectral components travels at different group velocities, causing the pulse energy to spread over time as the pulse propagates through the medium. Fiber group velocity dispersion is measured either in units of picoseconds squared per kilometer or picosecond per kilometer per nanometer. Roughly speaking a pulse with the bandwidth 1 nm spreads by corresponding number of ps over 1 km. Dispersion can be positive (this means that low frequencies travel at a higher speed than high frequencies) or negative (in this case high frequencies propagate at a higher speed than low frequencies). The dispersion of standard single mode fiber is positive (normal) for wavelengths shorter than 1300 nm and negative (anomalous) for wavelengths longer than 1300 nm. Standard monomode fiber (SMF) has dispersion of about 20 ps^2/km at wavelength 1550 nm. Corresponding dispersive spreading of 10 ps pulse in SMF after 125 km is about 50 ps or, in other words, 5 times its original width. Such a large spreading can lead to overlapping of neighboring bits and consequently to degradation of the information signal. Linear signal distortion caused by the GVD in fiber transmission systems can be almost suppressed by the dispersion compensation (mapping) technique. Optimisation of the system performance in the case of a linear transmission requires minimisation of the chromatic dispersion of the line. This can be achieved by operating close to the zero dispersion point or/and additional compensation of the accumulated dispersion. The idea to use a compensating fiber to overcome dispersion of the transmission one has been proposed in 1980 [12]. In the low power (linear) regime, compensation of dispersion aims to prevent dispersive broadening of the signal in the transmission fiber by the compression in the compensating fiber. An additional advantage is that the impact of the four-wave mixing on a signal transmission is suppressed due to the reduction of the efficiency of the phase matching. The dispersion compensation technique has been used successfully both in long-haul communication systems and in the existing terrestrial optical links, most of which are based on standard telecommunication fiber with large dispersion in the second optical window (at 1550 nm). The basic optical-pulse equalising system consists of a transmission fiber (i.e., standard monomode fiber in the installed links) and equaliser fiber with the opposite dispersion (e.g., dispersion compensating fiber (DCF) in the case of the transmission fiber with anomalous dispersion) [12].

In the linear (low power) systems dispersive broadening can be eliminated by dispersion compensation. However, the nonlinear effects can still be the primary reason for signal degradation especially in long-haul transmission systems. The response of the optical medium is not exclusively linear. The fiber refractive index instantaneously increases by an amount proportional to the optical power (Kerr effect). Modulation of the optical power leads to the corresponding modulation of the index. For instance, a high power light pulse increases the refraction index with corresponding change of the phase of the propagating pulse. This is the so-called self-phase modulation effect. The nonlinearity can delay the ‘fast’ spectral components relative the ‘slow’ carriers. As a result, the nonlinearity counterbalances the effect of dispersion broadening and the optical pulse becomes self-trapped. When nonlinearity and dispersion balance each other the pulse preserves its shape during propagation. Optical soliton was proposed by Hasegawa and Tappert in 1972 [13]. In 1971 Zakharov and Shabat demonstrated [14] integrability of the nonlinear Schrödinger equation – the basic mathematical model that somehow forms the theoretical background of the fiber optic communications. The first experimental observation of optical solitons in fiber have been realized in 1980 by Mollenauer et al. [15].

It is important to point out that one hardly can expect that the ideal soliton theories will be capable of describing in full practical real-world transmission systems. So why then do we think that DM soliton theory could be of any interest for system designers? Optimization of the optical transmission system parameters is a crucial task for the design of fiber links. Usually, time-consuming numerical simulations are required to find optimal operating regimes and optimal system parameters. Comprehensive investigation of stable regimes and their tolerance in multi-dimensional parameter space is limited by the computational time required for optimizations. The most natural characteristic of system performance is the bit-error-rate (BER). Direct numerical modeling of BER as small as required 10^{-9} is not feasible with currently available computer base. Therefore, it is of great interest to develop efficient and reliable indirect numerical and statistical methods to evaluate system performance. Limiting system analysis by consideration, for instance, of periodically recovered carrier only, one can gain instead a possibility of an advanced time-efficient optimization. Taking advantage of simple semi-theoretical methods well developed for DM solitons one can optimize parameters of the system and input signal before applying full numerical modeling.

3. Traditional optical soliton

In the general case, signal transformation along the fiber line is caused by a combined action of dissipation and amplification, dispersion and nonlinearity and cannot be described in a simple way. However, under certain conditions, stable soliton-like dynamics is possible even taking into account fiber loss. Under effect of periodic amplification solitons can undergo substantial power variations without losing their integrity.

Let us first recall the main features of *traditional* soliton fiber communication lines without (or with a weak) dispersion management. The design of long-haul lightwave communication systems assumes utilisation of periodically installed in-line erbium-doped fiber optical amplifiers to compensate a carrier signal attenuation in the transmission fiber. Average (slow) dispersive broadening of the pulse propagating in the anomalous dispersion region can be exactly compensated by the nonlinear phase shift. Thus, a traditional fundamental optical soliton relies on a balance between self-phase modulation and anomalous second-order dispersion that allows it to preserve the carrier signal shape over thousands of kilometers. Note that the positive use of the nonlinearity can also be achieved in a general return-to-zero (RZ) transmission with the rectangular carrier pulse occupying half of the bit period in the anomalous dispersion region. The important feature of such systems is that the amplifier spacing is considerably shorter than the characteristic dispersion and nonlinear lengths, and therefore, both the dispersion and the nonlinearity can be treated as perturbations on the scale of one amplification period. To leading order, only the fiber loss and the periodic amplification are significant factors affecting the pulse evolution between two consecutive amplifiers. These factors cause the power oscillations, while the form of the pulse remains approximately unchanged. On larger scales nonlinearity and dispersion come into play and the pulse propagation in such communication systems is described by the well established path-average (guiding-centre) soliton theory [16–18]. The average dynamics of the optical signal in this case is given to leading order by the integrable [14] NLS equation for the envelope of the electric field A:

$$\mathrm{i}\frac{\partial A}{\partial z} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + |A|^2 A = 0. \tag{1}$$

Here the typical problem to be addressed is a Cauchy problem: $A(0,t) = A_0(t)$ and we are interested in the evolution of the field $A(z,t)$ with z. Soliton solution of this equation has the well-known form

$$A(z,t) = \eta\,\mathrm{sech}[\eta t]\exp\left(\frac{\mathrm{i}\eta^2 z}{2}\right). \tag{2}$$

Integrability of the path-averaged model makes it possible to use well-developed mathematical techniques to analyse effects of numerous practical perturbations and boundary conditions. Solitons take advantage of their 'particle-like' nature. An important consequence of this feature is that the solitons are robust in the presence of various perturbations (such as loss, filtering, non-perfect launch conditions an so on) and the perturbed pulses will eventually evolve into stable solitons. The traditional soliton approach allows error-free transmission over transoceanic distances with 10 Gb/s and higher rates per channel [19]. Note that in recent years, impressive results have been achieved in high-bit-rate optical communications both using the linear transmission concept and the soliton-based optical signal transmission. The critical issue for soliton transmission is the soliton control. Traditional soliton transmission requires some form of the soliton control [20,21] to overcome an inherent limitations of this method. Major limitations on the conventional soliton transmission are timing (Gordon and Haus [22]) jitter, soliton interaction and four-wave mixing. Further progress in soliton communication systems has been achieved by using dispersion-management [23–67]. New results are reported continuously in this field and further rapid progress is expected, therefore, we do not aim here to present a comprehensive review of numerous works in this field or to overview the latest record experimental results. Instead, we will focus in the following sections on the general aspects of DM soliton theory.

4. Dispersion-managed solitons

Though dispersion management was applied originally in low-power (linear) transmission systems, it has been discovered recently that this technique is also a very promising way to increase transmission capacity of the soliton-based communication lines. A transmission line constructed from alternated fibers with anomalous or normal dispersion has a low path-averaged chromatic dispersion, but a high local one, thereby suppressing the Gordon–Haus timing jitter as well as the four-wave mixing efficiency simultaneously. In [23] it has been proposed to incorporate a section of dispersion compensating fiber into the standard periodic soliton transmission line, before each amplifier. This was perhaps the first formulation in the literature of the idea of dispersion-managed soliton transmission, even though a clear theoretical and practical description of this regime was not presented until few years later. It has been shown in [23] that this new (for the soliton systems) technique reduces the power required, compared to an uncompensated (constant dispersion) soliton system, and increases both the maximum transmission distance and the range of pulse width over which operation is possible. In the first related experimental work [24] it has been demonstrated that the dispersion management leads to a significant reduction of the Gordon–Haus timing jitter. It should be pointed out that a similar idea of stretched pulses generation in a loop (periodic) laser system with varying dispersion has been proposed in [28]. In [29,30] the dispersion-managed pulse has been identified as a new information carrier – a stable periodic breather with features very different from that of the conventional soliton. A path-average theory and a simple approximate method to describe breathing DM soliton dynamics resulting from the interplay between varying dispersion and nonlinearity has been developed in [30]. The importance of pulse chirping to reduce the chromatic dispersion penalty and to improve transmission capacity of DM systems has been pointed out in [43]. The energy of the dispersion-managed soliton is enhanced [29] in comparison with a fundamental soliton (soliton solution of the NLSE) of the same width corresponding to the same path-averaged dispersion. This energy enhancement is the important feature of DM soliton leading to the increase of the signal-to-noise ratio (SNR) with substantial improvement of the system performance [24]. Large variations of the dispersion (strong dispersion management) strictly modify the soliton propagation fundamentally, inducing breathing-like oscillations of the pulse width during the amplification period. This dynamics differs substantially from the path-averaged (guiding-centre) soliton propagation in systems with constant or weakly varying dispersion and from that of the traditional fundamental soliton (the soliton solution of the integrable NLS equation [14]). Nevertheless, numerical simulations and experiments have demonstrated that it is possible to observe extremely stable propagation of a breathing soliton in fiber links with strong dispersion management. Pulse dynamics presents rapid oscillations of the power and width on the amplification distance, and slow evolution on the larger scales due the fiber nonlinearity and residual dispersion [30]. Presumably the chirp is the most important feature of the dispersion-managed soliton [43,30]. Soliton chirp leads to a fast rotation of the relative phase shift between neighbouring solitons resulting in the suppression of interaction. An important consequence of the chirp of the dispersion-managed soliton is that the input signal launched into the transmission line should be either chirped [43,25] or launched at some specific points of the dispersion map (for the case of transform-limited input pulse) [32,67,59]. The most surprising feature of the DM soliton is that it can propagate stably along a transmission line with zero or even normal average dispersion, in contrast to the fundamental soliton that propagates stably only in the anomalous dispersion region [49]. This feature is extremely interesting, because the transmission of the finite energy pulse close to the zero dispersion point takes advantage of the suppressed timing jitter. Recall that the energy of the traditional fundamental soliton is proportional to the average chromatic dispersion. Therefore, to keep the signal-to-noise ratio large enough one has to operate not too close to the zero-dispersion point. On the other hand, a timing jitter resulting from the Gordon–Haus effect [22] is proportional to the fiber dispersion and it is preferable to transmit solitons at wavelengths close to the zero dispersion point. Thus, it would be desirable to produce a finite energy soliton pulse in the region of the low fiber dispersion. A possibility to transmit a DM soliton at very low average dispersion allows one to reduce timing jitter for a number of channels which is of a crucial importance for WDM transmission. All the listed features make clear the difference between breathing soliton-like pulse in a system with dispersion compensation and the soliton of the NLSE. As a particular consequence of this, an average model describing path-averaged (slow) evolution of the breathing DM pulse should differ from the NLSE (that governs path-averaged dynamics of the fundamental soliton). In other words, the dispersion management imposes such a strong perturbation that a carrier pulse in this case is no longer the NLSE soliton. In the case of a weak dispersion management a powerful Lie-transform technique has been applied to describe properties of the carrier pulse (dressed soliton) [50].

In Fig. 1 the evolution of a DM soliton during one section in normal (bottom) and logarithmic (up) scales is shown. The bottom figure demonstrates self-similar-like compression and recompression of the soliton. In the logarithmic scale it is seen that at the lines $z = 0$ and $z = 0.5$ there exist some dips corresponding to the points at which the soliton power $|A(z,t)|^2$ approaches zero (and consequently, the logarithm of the power tends to minus infinity). Note that the oscillations around the main peak cannot be completely suppressed because they present an inherent part of the DM soliton. Comprehensive mathematical theory of nonlinear wave propagation in systems with rapidly varying (including the principal case of large amplitude variations) dispersion has been presented in [30,67,50,60]. Numerical simulations and experiments have revealed the following main features of the DM soliton:

- the width and chirp (characteristic of the phase of the pulse) experience large oscillations during the compensation period leading to 'breathing-like' soliton dynamics;
- the shape of the forming asymptotic pulse is not always a sech shape as for the NLSE soliton, but varies with the increase of the strength of the map from a sech shape to a Gaussian shape and to a flatter waveform. The pulse shape varies along the compensation section from a monotonically decaying profile to a distribution with oscillatory tails;
- the time-bandwidth product varies with an increase of the map strength (that is a measure of the dispersive broadening proportional to the difference of the local dispersions times the fiber lengths and inversely proportional to the square of the pulse width) from 0.32 corresponding to the sech-shaped NLSE soliton to 0.44 corresponding to the Gaussian pulse and increases further with increase of the map strength;
- the energy of the stable breathing pulse is well above that of the NLSE soliton with the same pulse width and of the corresponding average dispersion;
- DM soliton can propagate at the zero path-average dispersion and even in the normal dispersion region;
- the central part of DM pulse is self-similar, but the far-field oscillating (and exponentially decaying) tails are not.

Recent developments in fiber-optic communications have demonstrated that dispersion management makes the features of soliton transmission close to those of non-soliton transmission [27,26,51]. It is already recognized that the dispersion-managed soliton presents a novel attractive type of a nonlinear carrier of information in optical fiber links. In the following sections we briefly overview the theory of DM solitons.

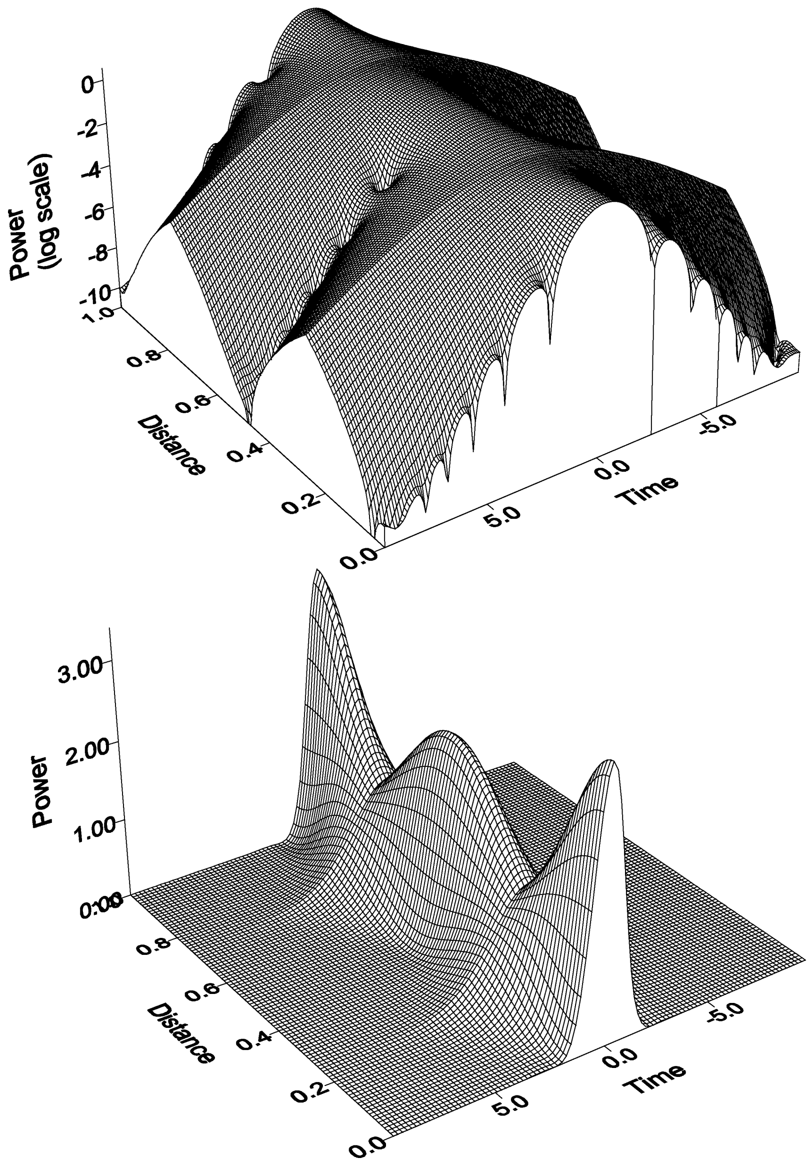

Fig. 1. Evolution over one compensation period of DM soliton shown in normal (lower) and logarithmic (upper) scales. In the leading order the dynamics is self-simlar (lower) and is given by Eqs. (20), (21). It is seen that the dips appear at the beginning (end) and in the middle of the considered symmetrical periodic cell. Dispersion $d(z) = \pm d + \langle d \rangle$ with $d = 5$ and $\langle d \rangle = 0.15$, power is normalized by $P_0 = 1$ mW.

5. Basic mathematical model and normalizations

The optical pulse propagation in a cascaded transmission system with varying dispersion is governed by

$$\mathrm{i}\frac{\partial E}{\partial z} - \frac{1}{2}\beta_2(z)\frac{\partial^2 E}{\partial t^2} + \sigma(z)|E|^2 E = \mathrm{i}\left[-\gamma(z) + r_k \sum_{k=1}^{N} \delta(z - z_k)\right] E = \mathrm{i}G(z)E.$$

Here z is the propagation distance in [km], t is the retarded time in [ps], $|E|^2 = P$ is the optical power in [W], β_2 is the first order group velocity dispersion measured in [ps^2/km]. We write σ, β_2 and γ as functions of z to account for the change of these parameters from fiber to fiber. It is customary to express the coefficient β_2 in terms of the associated dispersion parameter D by $\beta_2 = -\lambda_0^2 D/(2\pi c_l)$, where c_l is the speed of light and D is measured in ps/(nm·km). We denote the nonlinear coefficient by $\sigma = (2\pi n_2)/(\lambda_0 A_{\mathrm{eff}})$, where n_2 is the nonlinear refractive index, λ_0 is the carrier wavelength, A_{eff} is the effective fiber area,

Z_k are the amplifier locations. For simplicity, we consider below a periodic amplification with the period Z_a. If γ is constant between two consecutive amplifiers, then $r_k = [\exp(\gamma_k Z_a) - 1]$ is an amplification coefficient after the fiber span between the k-th and $(k-1)$-th amplifier. The loss coefficient $\gamma_k = 0.05\ln(10)\alpha_k$ accounts for the fiber attenuation along a fiber span before the k-th amplifier, where α_k is given in dB/km. Note that distributed amplification can be easily incorporated into the model through an appropriate gain/loss function $\gamma(z)$.

High local dispersion significantly changes pulse dynamics in comparison to systems with a constant group velocity dispersion even if the path-average dispersions are identical. A slow (average) dynamics on the large scales is determined by the effects of nonlinearity, residual (path-averaged) dispersion and average effects of the fast dynamics. It is customary to make the following transformation $E(z,t) = \sqrt{P_0}\,A(z,t)\exp(\int_0^z G(z')\,dz')$. The evolution of the optical signal envelope A along the cascaded fiber transmission system is then given by the NLS equation with periodic coefficients that can be written in the following form:

$$\mathrm{i}\frac{\partial A}{\partial z} + d(z)\frac{\partial^2 A}{\partial t^2} + \varepsilon c(z)|A|^2 A = 0. \tag{3}$$

We introduce here the following normalization: z is normalized to a length Z_0 (in km) defined below; time t is measured in some time constant t_0 (in ps) that can be specified for each specific problem; an envelope of the electric field E is normalized to the power parameter P_0: $|E|^2 = P_0\,|\,A\,|^2 \exp(2\int_0^z G(z')\,dz')$. Function

$$d(z) = \tilde{d}(z) + \langle d\rangle = -\frac{\beta_2(z)Z_0}{2t_0^2} = \frac{\lambda_0^2 Z_0 D(z)}{4\pi c_l t_0^2}$$

describes periodic compensation of dispersion (with the period L in physical units). In what follows we will use both physical and normalized dispersions. Periodic (with the period Z_a in real world units) function

$$\varepsilon c(z) = \tilde{c}(z) + \langle c\rangle = 2\pi\, n_2\, P_0 Z_0 \frac{\exp(\int_0^z G(z')\,dz')}{\lambda_0 A_{\mathrm{eff}}} = 2\pi n_2 P_0 Z_0 \frac{\exp[-2\gamma(z - z_k)]}{\lambda_0 A_{\mathrm{eff}}}$$

for $z_k \leqslant z < z_{k+1} = z_k + Z_a$ describes the power variation due to fiber loss and amplifier gain that is accounted through transformation of the pulse power at junctions corresponding to the locations of the optical amplifiers. The amplification distance Z_a in general can be different from the compensation period L. We consider a general case when L and Z_a are rational commensurable, namely, $n\,Z_a = m\,L = Z_0$ with integer n and m. This includes as particular limits all known and studied cases and allows us to describe a novel regime with short-scale ($L \ll Z_a$) management. The distance $z = Z/Z_0$ is normalised in Eq. (1) by a minimal common period Z_0 of the functions d and c and the averaging throughout the paper is over this period. In the normalised units 1-periodic d and c have basic periods $1/m$ and $1/n$ respectively. The small parameter ε is proportional to the pulse power. Eq. (1) possesses the conserved quantity $E = \int |A|^2\,dt$ which is related to the energy of the system. The optical pulse dynamics in the dispersion-managed transmission systems is determined by the combined action of the fiber loss and periodic amplification, self-phase modulation and varying chromatic dispersion. It should be pointed out that these effects are not additive and pulse evolution critically depends on the order in which dispersion compensation is realized. Strong interference of the effects of nonlinearity and varying dispersion leads to a rich variety of possible configurations for dispersion management.

There are two important limiting cases in modeling optical transmission with dispersion management. In the long-haul transmission system a period of the dispersion map can be much larger than the amplification distance $Z_a \ll L$. The inclusion of periodic amplification and dispersion compensation can be handled as separate problems, provided that the amplification distance is substantially different from the period of dispersion map [32]. Therefore, signal dynamics can be averaged over amplification period and an averaged propagation will be described in this limit by the lossless NLS equation with varying dispersion. We call this limit the *lossless model*. Obviously, this consideration includes fiber loss, but the average model formally is similar to a pulse propagation in lossless fiber. This justifies consideration of the lossless model that is as a matter of fact a problem with different scales of power and dispersion compensations. In the second limit, amplification period is of the order of the compensation period. The most simple and practical variant is to use a bobbin with compensating fiber just at the amplifier location station: $Z_a = L$. This limit is important in the problem of the upgrade of the existing terrestrial fiber links based on standard telecommunication fiber having high dispersion at 1550 nm.

6. Path-average theory of DM soliton

In this section, we overview results obtained in [30,37,40] where path-average DM soliton propagation equation in the spectral domain has been derived. Note that averaging can be performed by different methods that are essentially slightly different presentations of the same procedure: Hamiltonian averaging [37,41,42], Lie-transform [40], multi-scale expansion [35] and other methods. A comprehensive description (including calculations of the higher-order terms) of the averaging using Lie-transform can be found in [40].

Note that normal averaging cannot be applied to Eq. (1) directly because of a large variation of dispersion $d(z)$. Therefore, first, to eliminate the periodic dependence of the linear part we apply following [30,37] the so-called Floquet–Lyapunov transformation

$$A_\omega = \phi_\omega \exp\{-\mathrm{i}\omega^2 R_0(z)\}, \qquad \frac{\mathrm{d}R_0(z)}{\mathrm{d}z} = d(z) - \langle d\rangle. \tag{4}$$

Here $A_\omega = A(\omega, z)$ is a Fourier transform of $A(t,z) = \int A_\omega \exp[-\mathrm{i}\omega t]\,\mathrm{d}\omega$. In the new variables the equation takes the form

$$\mathrm{i}\frac{\partial \phi_\omega}{\partial z} = \langle d\rangle \omega^2 \phi_\omega - \varepsilon \int G_{\omega 123}\delta(\omega + \omega_1 - \omega_2 - \omega_3)\phi_1^* \phi_2 \phi_3 \,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3, \tag{5}$$

here $G_{\omega 123}(z) = c(z)\exp\{\mathrm{i}\Delta\Omega R_0(z)\}$ is 1-periodic and $\Delta\Omega = \omega^2 + \omega_1^2 - \omega_2^2 - \omega_3^2$. Note that $G_{\omega 123}$ depends only on the specific combination of the frequencies given by the resonance surface $\Delta\Omega$. Both the Fourier and the Floquet–Lyapunov transform (4) are canonical and the transformed Hamiltonian H can be written

$$H = \langle d\rangle \int \omega^2 |\phi_\omega|^2\,\mathrm{d}\omega - \varepsilon \int \frac{G_{\omega 123}}{2}\delta(\omega + \omega_1 - \omega_2 - \omega_3)\phi_\omega^* \phi_1^* \phi_2 \phi_3 \,\mathrm{d}\omega\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3. \tag{6}$$

Now we apply Hamiltonian averaging [37]. Let us make the following change of the variables

$$\phi_\omega = \varphi_\omega + \varepsilon \int V_{\omega 123}\delta(\omega + \omega_1 - \omega_2 - \omega_3)\varphi_1^* \varphi_2 \varphi_3 \,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3,$$

here $V_{\omega 123}(z) = \mathrm{i}\int_0^z [G_{\omega 123}(\tau) - T_{\omega 123}]\,\mathrm{d}\tau + \mathrm{i}V_{\omega 123}(0)$, $(\langle V_{\omega 123}\rangle = 0)$ with

$$T_{\omega 123} = \langle G_{\omega 123}\rangle = \int_0^1 c(z)\exp\{\mathrm{i}\Delta\Omega R_0(z)\}\,\mathrm{d}z. \tag{7}$$

In the leading order in ε, a path-averaged equation has the form (Gabitov and Turitsyn [30]):

$$\mathrm{i}\frac{\partial \varphi_\omega}{\partial z} = \langle d\rangle\, \omega^2\, \varphi_\omega - \varepsilon \int T_{\omega 123}\,\delta(\omega + \omega_1 - \omega_2 - \omega_3)\varphi_1^* \varphi_2 \varphi_3 \,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3. \tag{8}$$

This is the basic model of the DM soliton theory. This model has many interesting properties that have been discussed in [35, 37,41,42]. The corresponding averaged Hamiltonian H is

$$\langle H\rangle = \langle d\rangle \int \omega^2 |\varphi_\omega|^2\,\mathrm{d}\omega - \varepsilon \int \frac{T_{\omega 123}}{2}\delta(\omega + \omega_1 - \omega_2 - \omega_3)\varphi_\omega^* \varphi_1^* \varphi_2 \varphi_3 \,\mathrm{d}\omega\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3. \tag{9}$$

Note that derived path-averaged model can be used to describe slow (stroboscopic) evolution of arbitrary initial pulse not necessarily DM soliton. Fig. 2 shows a comparison of the spectral power distributions after transmission of input Gaussian signal over 8000 km found by path-average mapping (dashed line) and by direct numerical simulations of the full model (solid line).

The Hamiltonian averaging introduced here presents a regular way to calculate the next order corrections to the averaged model. From the Hamiltonian structure of the starting equation it is clear that the matrix element $T_{\omega 123}$ has the following symmetries $T_{\omega 123} = T_{1\omega 23} = T_{\omega 132} = T_{23\omega 1}^*$. Note that Eq. (8) possesses the remarkable property. The matrix element $T_{\omega 123} = T(\Delta\Omega)$ is a function of $\Delta\Omega$ and on the resonant surface $\omega + \omega_1 - \omega_2 - \omega_3 = 0$, $\Delta\Omega = \omega^2 + \omega_1^2 - \omega_2^2 - \omega_3^2 = 0$, both $T_{\omega 123}$ and its derivative over $\Delta\Omega$ are regular. This observation allows us to make the following quasi-identical-like transformation [37], which eliminates the variable part of the matrix element $T_{\omega 123}$

$$\varphi_\omega = a_\omega - \frac{\varepsilon}{\langle d\rangle}\int \frac{T_0 - T_{\omega 123}}{\Delta\Omega}\, a_1^*\, a_2\, a_3\, \delta(\omega + \omega_1 - \omega_2 - \omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3, \tag{10}$$

where $T_0 = T(0)$. This transformation has no singularities. If the integral part in this transform is small compared with a_ω, then in the leading order we get for $a(z,t)$ NLS equation. Obviously, this transformation is quasi-identical only if the integral in Eq. (10) is small compared with a_ω. This is not true in a general case and that is why, typical solution of Eq. (8) has a form [35] different from cosh-shaped NLSE soliton. However, if the kernel function in Eq. (10) is small

$$\bigl|S(\Delta\Omega)\bigr| = \left|\frac{T_0 - T_{\omega 123}(\Delta\Omega)}{\Delta\Omega}\right| \ll 1, \tag{11}$$

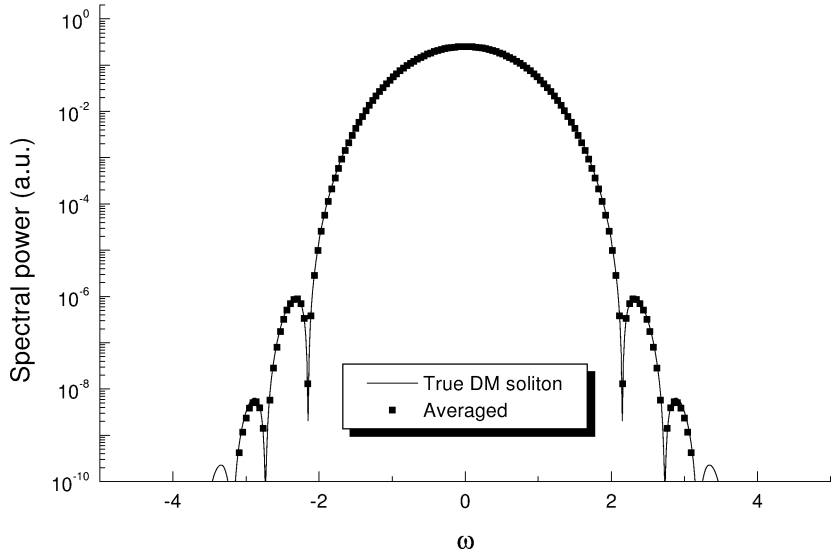

Fig. 2. Comparison of the spectral power distributions after transmission over 8000 km found by path-average mapping (dashed line) and by direct numerical simulations of the full model (solid line). Input Gaussian signal with the peak power 1 mW and width 33 ps propagates in the system composed from fibers with $D_1 = 2.6$ ps/nm/km and $D_2 = -2.2$ ps/nm/km and the length 200 km each. Amplification distance is 50 km ($L/Z_a = 4$).

then the averaged model can be reduced to the NLSE. In other terms, this is a condition on the functions $c(z)$ and $d(z)$ that makes possible quasi-identical transformation.

Let us now demonstrate how developed above general theory can be applied to specific physical problems. First consider the case of $L \geqslant Z_a$ that is typical, for instance, for transoceanic lines. To be specific, let us analyze, as an example, two-step dispersion map with the amplification distance Z_a km and dispersion compensation period $L = 2M \times Z_a$ km $= Z_0$ km ($m = 1,\ n = 2M$). Dispersion $d(z) = d + \langle d\rangle$ if $0 < Z < M \times Z_a = L/2$ and $d(z) = -d + \langle d\rangle$ if $M \times Z_a < Z < 2M \times Z_a = L$. The mean-free function R defined above can be found as $R(z) = d\,z - d/4$ if $0 < z < 1/2$ and $R(z) = -d[z - 1/2] + d/4$ if $1/2 < z < 1$. After some calculations, it can be found that the kernel of the function $T_{\omega 123} = T(\Delta\Omega)$ in such a system is

$$T(X) = \frac{G-1}{G\ln G}\frac{\sin[XM]}{M}\frac{1}{(1+[2X/\ln G]^2)}\left\{\frac{\cos[X]}{\sin[X]} + \frac{2X}{\ln G}\frac{G+1}{G-1}\right\}, \qquad (12)$$
$$X = \frac{\Delta\Omega\, Z_a\, d}{2L} = \frac{\Delta\Omega d}{4M}.$$

Here the gain $G = \exp[2\gamma Z_a]$ (γ is a fiber loss). It is interesting to look at some particular limits in this general formula. First, if $d = 0$ (uniform dispersion along the system) we reproduce the result of Mollenauer et al. [16–18]: $T(\Delta\Omega) = (G-1)/(G\ln G)$ and because T is a constant, path-averaged model is just the integrable NLS equation. The second limit is the so-called 'lossless' model [29] ($\gamma = 0$). In this case $T(\Delta\Omega) = \sin[\Delta\Omega\, d/4]/[\Delta\Omega\, d/4]$. We justify now the use of the 'lossless' system [29] for modeling of the practical (with fiber loss) fiber transmission system. It is interesting that the theory developed here confirms that the periodic amplification and dispersion compensation can be handled as separate problems, provided that amplification distance is substantially different from the period of dispersion map. For $M > 1$ function $T(\Delta\Omega)$ is getting close and close to that one for the 'lossless' model $T(\Delta\Omega) = \sin[\Delta\Omega\, d/4]/[\Delta\Omega\, d/4]$ multiplied by the path-averaged factor $(G-1)/(G\ln G)$. The result obtained proves that the power budget and the dispersion mapping, effectively, can be handled separately in long-haul transoceanic optical communication systems where amplification distance is typically much shorter than the dispersion compensation period.

Next we consider a relatively recently proposed regime with a short-scale ($L \ll Z_a$, and a general case $L \leqslant Z_a$) dispersion management. Optical fibers with $L \ll Z_a$ have recently been manufactured [38]. Recall that ultra-short, power-enhanced DM solitons in the traditional systems with $L \geqslant Z_a$ typically have too high power to be realized in practice. It is of interest to find stable propagation regimes with short and low power solitons. As has been shown in [39] rather short ($\leqslant$5 ps) DM solitons in systems with the short-scale dispersion management could have low enough energy to provide for stable ultra high-bit-rate ($\geqslant$40 Gb/s per channel) transmission. Here we demonstrate that a path average propagation in systems with a short-scale management $L \ll Z_a$ (even with the *large variations* of the dispersion) can be described by the integrable NLS equation. Again to be specific, let us consider a two-step dispersion map with the amplification distance $Z_a = Z_0$ ($n = 1$) and dispersion compensation period $L = Z_a/m$ km (or $1/m$ in the normalized units). Normalized dispersion $d(z) = d + \langle d\rangle$ if $k/m < z < (k+0.5)/m$ and $d(z) = -d + \langle d\rangle$ if $(k+0.5)/m < z < (k+1)/m$, here $k = 0, 1, 2, \ldots, m-1$. Mean-free function R defined above can be found as

$$R(z) = d\left(z - \frac{k}{m}\right) - \frac{d}{4m} \quad \text{if } \frac{k}{m} < z < \frac{k+0.5}{m} \quad \text{and} \quad R(z) = -d\left[z - \frac{k}{m} - \frac{1}{2m}\right] + \frac{d}{4m} \quad \text{if } \frac{k+0.5}{m} < z < \frac{k+1}{m}.$$

After some calculation, it can be shown that the matrix element $T_{\omega 123}$ in such a system is

$$T_{\omega 123} = T(Y) = \frac{G-1}{G\ln G}\frac{1}{1+(4mY/\ln G)^2}\left\{\exp(-\mathrm{i}Y) + \frac{4mY}{\ln G}\left[\sin Y\,\frac{G^{1/(2m)}+1}{G^{1/(2m)}-1} + \mathrm{i}\cos Y\,\frac{G^{1/(2m)}-1}{G^{1/(2m)}+1}\right]\right\}. \tag{13}$$

Here $Y = d\Delta\Omega/(4m)$. Next we estimate the matrix element of the quasi-identical transformation

$$\left|S(\Delta\Omega)\right| \leqslant \left|\int_0^1 \frac{c(z)[\exp(\mathrm{i}\Delta\Omega R(z)) - 1]}{\Delta\Omega}\,\mathrm{d}z\right| \leqslant \int_0^1 \left|c(z)R(z)\right|\mathrm{d}z \leqslant \max(R)\langle c\rangle = \frac{\langle c\rangle d}{4m}.$$

One can see that with increase of m (for the fixed other parameters) the path-averaged model (8) governing DM soliton propagation converges to the integrable NLS equation. Fig. 3 shows the power of the DM solitons (solid line) and the fundamental soliton with the same amplitude (dashed line) for different values of m. It can be seen that for large m the form of the DM solitons is very close to the shape of the fundamental soliton. It is interesting to note that in the limit of a very short-scale management (large m) we again get for T the lossless model approximation multiplied by the factor

$$\frac{G-1}{G\ln G} : T(Y) = \frac{\sin[\Delta\Omega\, d/(4m)]}{\Delta\Omega\, d/(4m)}\,\frac{G-1}{G\ln G}.$$

However, increase of m (decrease of L) under the fixed characteristic bandwidth of the signal makes insignificant oscillatory structure of the kernel. This means that if $T(Y)$ is practically concentrated in some region ΔY, then for large m corresponding

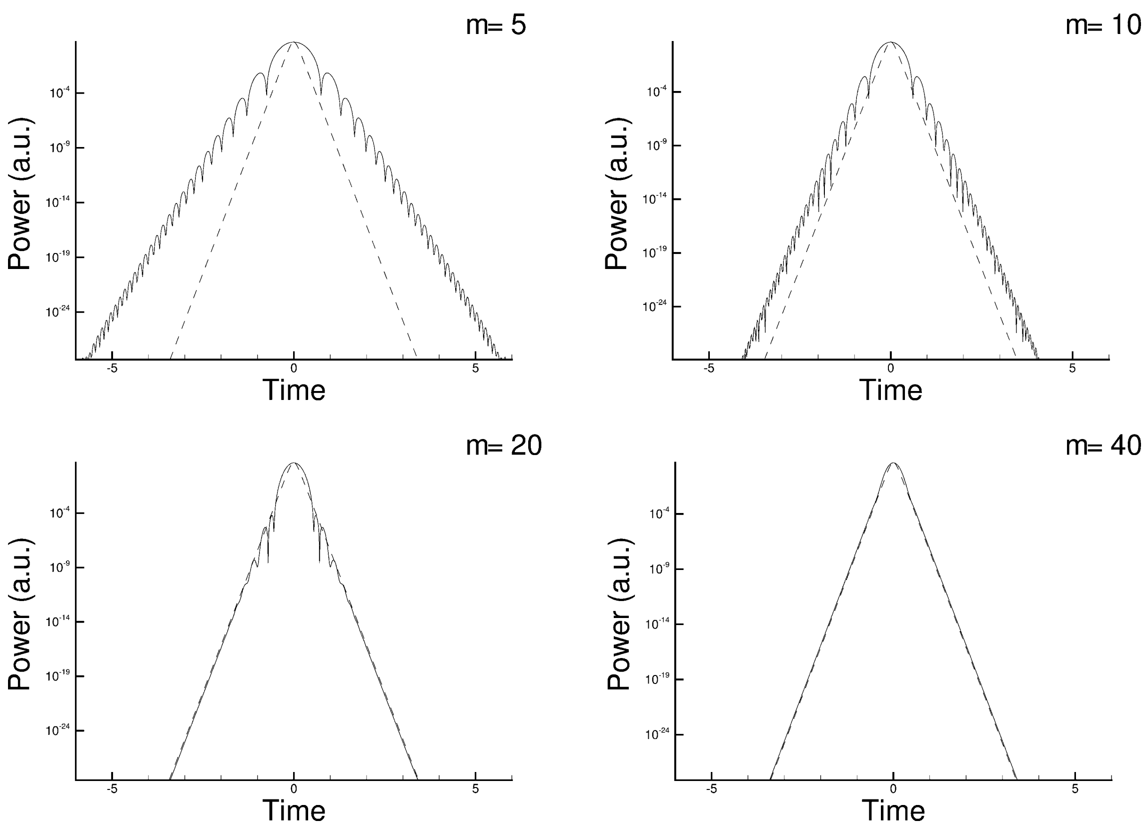

Fig. 3. The power of the DM solitons (solid line) and the fundamental soliton with the same amplitude (dashed line) for different m.

region in $d\,\Delta\Omega$ will be larger than for small m. For the pulses with the same spectral width this will mean that T is much flatter for large m and, as a matter of fact, for large m (small L) function T can be better approximated by a value $T(0)$. As a result, NLSE model works rather well in this limit and solution (of the path-averaged model!) should be close to cosh-like soliton of the NLSE. Of course, the function $|S(Y)|$ increases with the growth of L and is not small in the opposite limit $Z_a \ll L$ (lossless model). Therefore, the shape of DM soliton in the lossless model [29] (and large variations of the effective dispersion) is not cosh as it is in the considered model. In contrast to the lossless model, evolution of soliton parameters over one period is highly asymmetric here due to loss. Rapid variations of the pulse width, peak power and chirp are accompanied by the exponential decay of the power due to loss. Nevertheless, numerical simulations have revealed that there exists a true periodic solution that reproduces itself at the end of the compensation cell (in this case – at the end of the amplification period). Note that though it is known that for the lossless model in the so-called weak map limit [29,30,49,45] the DM soliton has a shape close to cosh, this is not so obvious for system with loss and different periods of amplification and dispersion variations.

7. Root-mean-square momenta method

An advantage of the transmission of the soliton carrier signal (in general, RZ formatted data) is that it can be described by a few main parameters, such as pulse width, peak power, chirp parameter and spectral width (the latter can be expressed through pulse width and chirp parameter). The particle-like behavior of the solitary wave signal allows one to make use of well developed mathematical methods to understand features of a such carrier and to predict effects that occur due to practical boundary conditions and due to deviations of real fiber properties from an ideal model. In the integrable and near-integrable models evolution of these few main soliton parameters can be calculated using perturbation methods. In the general case some information can be gained by considering evolution of the integral quantities – different root-mean-square momentum [48]. In this subsection we present a generalised momentum method to describe the main RMS DM soliton characteristics. Here we briefly overview results obtained in the papers [48,55,56]. This simple and transparent method is very useful in the modeling of an arbitrary dispersion-managed fiber links that typically involves many free parameters to be optimised. To describe rapid self-similar dynamics of the main peak of DM pulse let us first consider following [48] the evolution of the integral quantities related to the pulse characteristics: root-mean-square (RMS) pulse width T_{int}, pulse power chirp $M_{\text{int}}/T_{\text{int}}$

$$T_{\text{int}}(z) = \left[\frac{\int t^2|A|^2\,\mathrm{d}t}{\int |A|^2\,\mathrm{d}t}\right]^{1/2}, \tag{14}$$

$$\frac{M_{\text{int}}(z)}{T_{\text{int}}(z)} = \frac{\mathrm{i}}{4}\frac{\int t\,(AA_t^* - A^*A_t)\,\mathrm{d}t}{\int t^2|A|^2\,\mathrm{d}t}. \tag{15}$$

It is easy to check that the evolution of $T_{\text{int}}(z)$ and $M_{\text{int}}(z)$ is given by

$$\frac{\mathrm{d}T_{\text{int}}}{\mathrm{d}z} = 4d(z)M_{\text{int}}(z), \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(T_{\text{int}}M_{\text{int}}) = d(z)\Omega_{\text{RMS}}^2 - \frac{\varepsilon c(z)}{4}P_{\text{RMS}}. \tag{17}$$

Here we introduce

$$P_{\text{RMS}}(z) = \frac{\int |A|^4\,\mathrm{d}t}{\int |A|^2\,\mathrm{d}t}, \qquad \Omega_{\text{RMS}}^2 = \frac{\int |A_t|^2\,\mathrm{d}t}{\int |A|^2\,\mathrm{d}t}. \tag{18}$$

Integrating Eq. (17) over one period we get a simple explanation of why DM solitons can propagate at the zero and normal average dispersion. Indeed, for when T_{int} and M_{int} are exactly periodic, we have

$$\big\langle d(z)\Omega_{\text{RMS}}^2(z)\big\rangle = \frac{\varepsilon}{4}\big\langle c(z)\,P_{\text{RMS}}(z)\big\rangle. \tag{19}$$

When d and c are constant this gives us the RMS power of the soliton. If d is constant but $c(z)$ varies periodically this equation gives the guiding-centre (path averaged) enhancement factor. Finally, when both d and c are periodic functions of z (or c is constant as in lossless model) one can see that the requirement $\langle d\rangle > 0$ for the existence of conventional solitons is replaced by a condition $\langle d(z)\,\Omega_{\text{RMS}}^2\rangle > 0$, which can be satisfied even when the average dispersion is zero or positive as it is seen in Fig. 4. Different lines in Fig. 4 corresponds to different values of $T_{\text{int}}(0)$: 0.75 (short-dashed line), 0.65 (dashed line) and 0.55 (solid line). The same formula explains the power enhancement of DM soliton as well. To obtain a closed system of equations on T_{int} and M_{int} one has to derive equations on Ω_{RMS} and P_{RMS} and to express all intermediate integrals that appear in such equations in terms of the introduced above RMS momenta. This is possible only under some additional assumptions about the structure of

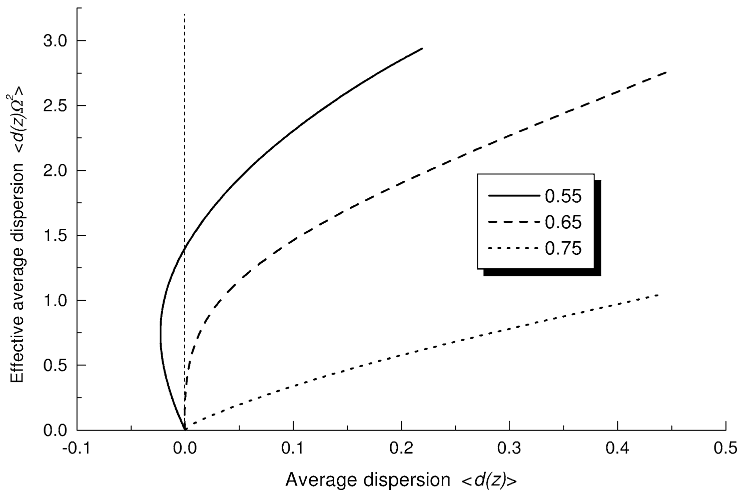

Fig. 4. Effective dispersion $D_{\text{eff}} = \langle d(z)\Omega^2(z)\rangle$ found from Eqs. (20), (21) ($\Omega(z)$ is the spectral bandwidth of the pulse) versus path-averaged dispersion. Solid line: $T(0) = 1$, bold dashed line $T(0) = 0.8$, dotted line $T(0) = 0.7$. The same map as in Fig. 1, but with different average dispersions.

the solution. Note, that the chirp (a first time derivative of the phase taken at the point where pulse amplitude is maximal) of the typical DM pulses shows a linear behavior in the region where most of the energy is concentrated. Of course, this is only a first approximation of the more complex phase picture and chirp is not at all linear in the whole time domain. However, note that the phase dependence appears in the necessary integral formulas only in the constructions like $|A|^4(\arg(A))_{tt}$ being multiplied by $|A|^4$ or other powers (like $|A|^2(\arg(A))_t$) of a fast decaying function $|A|^2$. Therefore, the contribution in the integral pulse characteristics due to deviations from the parabolic (in time) law in the phase is negligible in many practical situations (with highly localised $|A|^2$). Therefore, one can take a parabolic approximation of the phase near the pulse peak power location $(\arg(A)(z,t) = \phi(z)\,t^2 + \phi_0(z)$. The evolution equations on Ω_{RMS} and P_{RMS} then can be written in a simple form

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\Omega^2_{\text{RMS}}\right) = -2\varepsilon c(z)\phi(z)P_{\text{RMS}}, \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}P_{\text{RMS}} = -4d(z)\phi(z)P_{\text{RMS}}. \tag{21}$$

We have now four equations for the five quantities T_{int}, M_{int}, Ω_{RMS}, P_{int}, and ϕ. The missing last relation that is necessary to obtain a closed system of equations is given by:

$$\frac{M_{\text{int}}(z)}{T_{\text{int}}(z)} = \frac{1}{2}\frac{\int |A|^2 t(\arg(A))_t\,\mathrm{d}t}{\int t^2|A|^2\,\mathrm{d}t} = \phi(z). \tag{22}$$

Thus, the closed system of RMS momentum equations is presented by Eqs. (16), (17), (20), (21) and (22).

Derived RMS equations can be transformed to the basic model of two ODEs first obtained in the context of the cascaded fiber transmission lines by Gabitov and Turitsyn in [31] using a variational approach. To do this, note that equations on P_{RMS} and Ω_{RMS} can be (after simple manipulations) integrated as

$$P_{\text{RMS}}(z)T_{\text{int}}(z) = P_{\text{RMS}}(0)T_{\text{int}}(0) = 4\,const_1. \tag{23}$$

$$\left[\Omega^2_{\text{RMS}}(z) - 4M^2_{\text{int}}(z)\right]T^2_{\text{int}}(z) = const_2. \tag{24}$$

Substitution of

$$\Omega^2_{\text{RMS}}(z) = 4M^2_{\text{int}}(z) + \frac{const_2}{T^2_{\text{int}}(z)} \tag{25}$$

and Eq. (23) into Eq. (17) yields equations in T_{int} and M_{int} (see [31,48,56]):

$$\frac{\mathrm{d}T_{\text{int}}}{\mathrm{d}z} = 4d(z)M_{\text{int}}(z), \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}M_{\text{int}} = \frac{d(z)const_2}{T^3_{\text{int}}} - \frac{\varepsilon c(z)const_1}{4T^2_{\text{int}}}. \tag{27}$$

An advantage of the approach presented here is that it used only *one* assumption about the structure (phase) of the DM pulse to derive these basic equations.

8. DM soliton expansion in the basis of the Gauss–Hermite functions

In this section following [52–54,67] we present a rigorous mathematical method to describe the breathing dynamics of both the self-similar core and of the oscillating tails of a DM soliton. In other terms, this method will allow one to estimate accurately the deviations of a true DM soliton from the self-similar structure assumed in the RMS momentum description presented above. The dynamics of the DM soliton can be presented as a self-similar evolution of the main peak accompanied by oscillations of far-field tails that have non-self-similar structure [49]. Though the energy of the tails is much smaller compared with the energy of the central peak they are responsible for the non-self-similar periodic change of the pulse form during the evolution along the compensation cell [49]. An arbitrary input pulse propagating down the dispersion-managed line typically evolves into an asymptotic structure that presents self-similar rapidly oscillating main peak and a dispersive pedestal [33]. By a proper choice of the parameters of the input pulse this radiation can be significantly suppressed. However, oscillatory far-field tails around the main peak cannot be entirely suppressed, because they present an unalienable part of the DM soliton. Using an orthogonal set of chirped Gauss–Hermite functions one can derive path-averaged equation governing slow evolution and the shape of the DM soliton. In this way, one can obtain a set of ordinary differential equations for the coefficients of expansion of DM soliton in terms of chirped Gauss–Hermite functions. A generalised solution of the propagation equation with arbitrary input pulse can be presented in terms of chirped Gauss–Hermite orthogonal functions. This approach can be also useful in numerical modeling of the dynamics of arbitrary initial signal in the dispersion-managed communication systems.

There is an interesting analogy between a DM soliton and the nonlinear macroscopic quantum oscillator. The basic idea is that the periodic variations of the phase (that occur due to periodic oscillations of the dispersion) create an effective parabolic trapping potential. Without nonlinearity any propagating wave is a direct combination of the eigenfunctions of such a quantum oscillator potential – the Gauss–Hermite functions. When nonlinearity comes into play, the energy is redistributed between different modes. Therefore, a DM soliton can be viewed as a ground state of an effective macroscopic nonlinear quantum oscillator.

To remove from Eq. (3) the rapid self-similar dynamics that occurs due to large variations of the local dispersion let us apply the following self-similar transformation [67]

$$A(z,t) = \frac{N\,\mathrm{e}^{\mathrm{i}(M(z)/T(z))t^2}}{\sqrt{T(z)}} \sum_{n=0}^{n=\infty} B_n(z) f_n\left[\frac{t}{T(z)}\right] \mathrm{e}^{\mathrm{i}\lambda_n R(z)}, \tag{28}$$

here $x = t/T(z)$. The rapid oscillations of pulse width and chirp are accounted by periodic functions $T(z)$ and $M(z)$ and slow evolution is given by $f_n(x)$. We choose here the periodic functions T and M to keep in the leading order the self-similar structure of the DM pulse. We define equations on the functions T and M to be the same as for the integral quantities T_{int} and M_{int} in the section above:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = 4d(z)M, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{d(z)}{T^3} - \frac{\varepsilon c(z)N^2}{T^2}. \tag{29}$$

Here $T(L) = T(0)$, $M(L) = M(0)$ and N is a constant to be determined from the requirement that T and M are periodic solutions of Eq. (29). The functions $f_n(x)$ are the orthogonal normalised Gauss–Hermite functions:

$$f_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp\left(-\frac{x^2}{2}\right) H_n(x), \tag{30}$$

$$(f_n)_{xx} - x^2 f_n = \lambda_n f_n, \quad \lambda_n = -1 - 2n. \tag{31}$$

Here $H_n(x)$ is the n-th-order Hermite polynomial and coefficients B_n are given by the ordinary scalar product in $\mathcal{L}^2$ with f_m and $\mathrm{d}R/\mathrm{d}z = d(z)/T^2(z) - \langle d/T^2 \rangle$.

Inserting this expansion into basic equation (3) after scalar multiplication with f_m we obtain a system of ordinary differential equations for the coefficients B_n.

$$\mathrm{i}\frac{\mathrm{d}B_n}{\mathrm{d}z} + \left\langle \frac{d}{T^2} \right\rangle \lambda_n B_n + \beta(z) \sum_{m=0} \mathrm{e}^{2\mathrm{i}(n-m)R(z)} S_{n,m} B_m + \beta(z) \sum_{m,l,k} \mathrm{e}^{2\mathrm{i}(n+k-m-l)R(z)} B_m B_l B_k^* V_{m,l,k,n} = 0. \tag{32}$$

Here we introduce the notation

$$
\begin{aligned}
S_{n,m} &= \langle f_m | x^2 f_n \rangle = \int_{-\infty}^{+\infty} f_m(x) x^2 f_n(x)\,\mathrm{d}x, \\
V_{n,m,l,k} &= \langle f_m | f_n f_l f_k \rangle = \int_{-\infty}^{+\infty} f_n(x) f_m(x) f_l(x) f_k(x)\,\mathrm{d}x.
\end{aligned} \tag{33}
$$

Since integrals of the form $\int x^n \mathrm{e}^{-\alpha x^2}$ can be calculated analytically, it is possible to determine any $S_{n,m}$ and $V_{n,m,l,k}$. Symmetrical integrals $S_{n,m} = S_{m,n}$ are: $S_{n,n} = n + 0.5$, $S_{n,(n+2)} = 0.5\sqrt{(n+2)(n+1)}$. The other $S_{n,m}$ are zero if $m > n$.

Eq. (32) can be averaged directly (in contrast to the master equation (3)) because the large variations of the dispersion are moved to the phase factor proportional to $R(z)$. Averaging can be performed either using Lie-transform technique [60] or Hamiltonian averaging [55]. However, the most important zero-order term can be obtained directly. Let us split B_n into slow (U_n) and fast (η_n) varying parts $B_n = U_n + \eta_n + \cdots$ ($\mathrm{d}\eta_n/\mathrm{d}z \gg \eta_n$) and we assume also that rapidly varying part is small compared with slow varying one $\eta_n \ll U_n$. Averaging over one period in the leading order then gives for U_n

$$
\mathrm{i}\frac{\mathrm{d}U_n}{\mathrm{d}z} + \left\langle \frac{d}{T^2} \right\rangle \lambda_n U_n + \sum_{m=0} \left\langle \beta(z)\,\mathrm{e}^{2\mathrm{i}(n-m)R(z)} \right\rangle S_{n,m} U_m + \sum_{m,l,k} \left\langle \beta(z)\,\mathrm{e}^{2\mathrm{i}(n+k-l-m)R(z)} \right\rangle U_m U_l U_k^* V_{n,m,l,k} = 0. \tag{34}
$$

The steady-state solution of this path-averaged equation having the form $U_n = F_n \exp(\mathrm{i}kz)$ with F_n non-dependent on z presents DM soliton for given dispersion map. The derived equation permits one to describe in a rigorous way properties of DM solitons and more generally the propagation of any input signal for arbitrary dispersion map. Considering a solution in the form $U_n = F_n \exp(\mathrm{i}kz)$ we obtain the expansion of the DM soliton in terms of chirped Gauss–Hermite functions. The shape of any DM soliton can be found from a solution of the equation on F_n:

$$
-kF_n + \left\langle \frac{d}{T^2} \right\rangle \lambda_n F_n + \sum_{m=0} \left\langle \beta(z)\,\mathrm{e}^{2\mathrm{i}(n-m)R(z)} \right\rangle S_{n,m} F_m + \sum_{m,l,k} \left\langle \beta(z)\,\mathrm{e}^{2\mathrm{i}(n+k-l-m)R(z)} \right\rangle F_m F_l F_k^* V_{n,m,l,k} = 0. \tag{35}
$$

Note that though this nonlinear eigenvalue problem looks not simple at all, this is a set of *algebraic* equations that are much easier to solve than to find DM soliton from original PDE (3). Rapid convergence, that is natural for bell-shaped pulse means that localised pulse will be well represented by a limited number of terms in the expansion. This makes such basis very useful

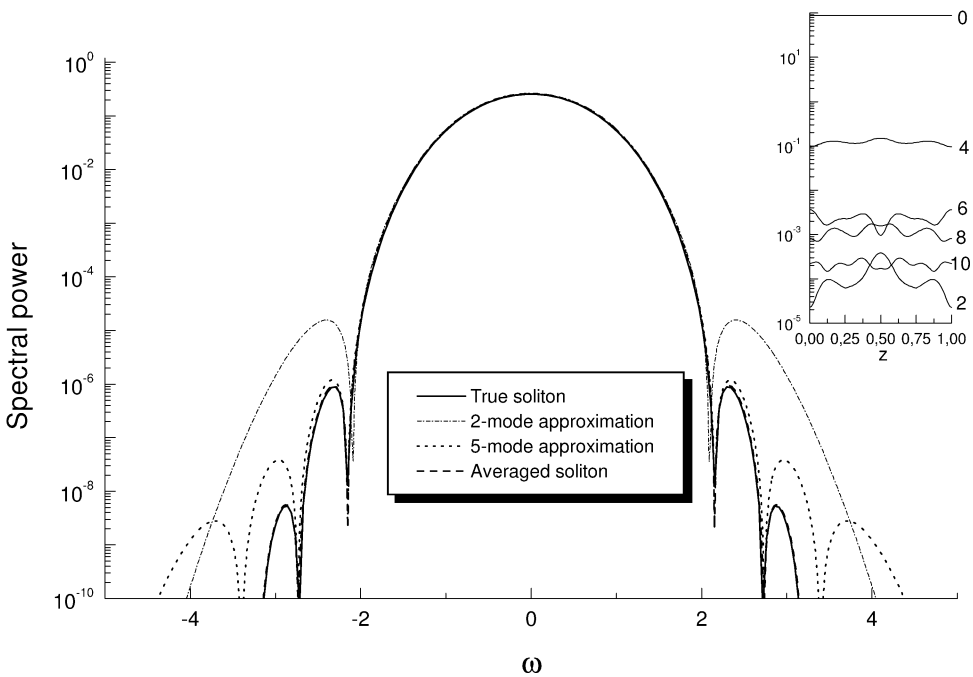

Fig. 5. Spectral power of DM soliton is shown in the logarithmic scale: true DM (solid line), solution of the path-averaged equation (8) (dashed line), two-mode (dashed-dotted line) and five-mode (dotted line) approximations using expansion of the soliton the the basis of the Gauss–Hermite functions. Inset shows dynamics of the first nonzero coefficients in this expansion over one period.

in different practical applications. This approach is a rigorous way to describe a family of DM soliton for an arbitrary dispersion map. A complete set of orthogonal chirped Gauss–Hermite functions is very useful in numerical simulations evolution of arbitrary shaped initial signal in DM fiber systems. In Fig. 5 it is presented a comparison of the path-average models presented in the paper with direct numerical simulations for two-step dispersion map (as in Fig. 1). Spectral power (logarithmic scale) of true DM soliton (solid line), taken at the boundary between two fibers is compared with two-mode $(0+4)$ (dashed-dotted line) and five-mode (dotted line) approximations in the expansion using the chirped Gauss–Hermite functions and solution of the path-averaged equation (8) (dashed line). Inset shows dynamics over one period of the first nontrivial (nonzero) coefficients in the Gauss–Hermite expansion of DM soliton. It is seen that both path-average models give quite good approximation of the true DM soliton. Even two-mode $(0+4)$ approximation describes the central part very well. Expansion in the basis of the Gauss–Hermite functions present analytical approximation of the DM soliton, describing both the Gaussian core and the oscillating tails.

9. Conclusions

We have overviewed in this article main physical and mathematical properties of dispersion-managed optical solitons. It is apparent that the development of DM soliton-based fiber communications enters the stage of commercial exploitation and possibility of real-world soliton networks. Wavelength-division-multiplexing transmission of DM solitons is an attractive way to realize long-distance ultra-high capacity fiber communication systems and to upgrade existing fiber networks to terabit per second regimes. We have presented here the theory of DM solitons at two levels of mathematical accuracy. Consideration of root-mean-square momenta, or equivalently, main order expansion of DM soliton in the Gauss–Hermite basis leads to simple set of ordinary differential equations governing evolution of the soliton width and chirp. This computer time saving approach can be especially useful for optical system optimization. More careful path-average theory is capable to describe in detail both a fine structure of DM soliton form and its evolution along the fiber line.

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