On the theory of the modulation instability in optical fiber amplifiers

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The modulation instability (MI) in optical fiber amplifiers and lasers with anomalous dispersion leads to cw radiation breakup. This can be both a detrimental effect limiting the performance of amplifiers and an underlying physical mechanism in the operation of MI-based devices. Here we revisit the analytical theory of MI in fiber optical amplifiers. The results of the exact theory are compared with the previously used adiabatic approximation model, and the range of applicability of the latter is determined.

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Modulation instability (MI) is a fundamental nonlinear effect [1–3] that manifests itself in optics, e.g., as a spontaneous breakup of a cw radiation with high enough power into a modulated light wave or a periodic train of pulses (it is not possible to overview all the literature on MI effects in optics; however, here we focus only on MI in active optical media; see, e.g., [1–11] and references therein). In active fiber, the MI is enhanced when interactions with optical noise provide seeding perturbations over a range of wavelengths [3]. Amplification of powerful laser radiation in optical fiber amplifiers with anomalous dispersion might suffer from the MI effect [3] that leads to the breakup of cw radiation and the appearance of multiple pulses. MI may impact the performance of an Er-doped amplifier in the case of signal bandwidth larger than the Brillouin gain width (35 MHz), when stimulated Brillouin scattering is suppressed. However, MI can also be exploited in a constructive way, for instance, as a technique to generate an optical pulse train or as a passive mode-locking mechanism in fiber lasers [4–8]. Recent progress in microstructured optical fibers offers new opportunities for the control of dispersive properties and thus to new potential applications of MI across a broad spectral range. Quantitative analysis of the MI is important for design and optimization of fiber lasers and amplifiers in which the wave intensity grows exponentially and MI dramatically intensifies nonlinear instabilities. As we show below, despite a number of publications, some important aspects of the MI development over finite device distances have not yet been comprehensively studied.

Over a wide range of physical parameters, propagation of the optical field down a fiber amplifier at leading order is described by the equation

$$i\frac{\partial \Psi}{\partial z} - \frac{\beta_2}{2}\Psi_{tt} + \gamma|\Psi|^2\Psi = i\frac{g_0}{2}\Psi + i\frac{g_0 T_2^2}{2}\Psi_{tt}. \qquad (1)$$

Here β_2 is the group-velocity dispersion, nonlinear parameter $\gamma = 2\pi n_2/(\lambda_0 A_{\rm eff})$ (λ_0 is the operational wavelength, n_2 is the nonlinear refractive index, and $A_{\rm eff}$ is the effective area of the fiber), and g_0 is the small signal gain of the amplifier. The parameter T_2 characterizes the gain bandwidth (or effect of external filtering). An optical field propagates here from $z=0$ to $z=L$. Consider the MI of the cw field, $\Psi(z,t) = (\sqrt{P_0} + a + ib)\times \exp[\frac{g_0 z}{2} + iP_0\int\gamma(z')dz']$; here $\gamma(z) = \gamma(0)\exp[g_0 z]$. Perturbation to the power evolution then can be found as $|\Psi(z,t)|^2 = (P_0 + 2a(z,t)\sqrt{P_0} + a^2 + b^2)\times\exp[g_0 z]$. Assuming a, $b \ll \sqrt{P_0}$, and expressing the fields a, b through the corresponding Fourier modes a_ω, $b_\omega \propto \exp[-i\omega t]$ (we omit in what follows the index ω) yields the standard linear evolution equations for the spectral modes of perturbations with the initial conditions to the Caushy problem $a(0)$, $b(0)$. When $\gamma = \text{const}$, $T_2 = 0$, $a \propto \exp[ik_z z]$ leads to the standard MI relation [1] $k_z^2 = \frac{\beta_2\omega^2}{2}[\frac{\beta_2\omega^2}{2} + 2\gamma P_0]$, with k_z increasing for small values of ω, reaching its maximum at $\omega_{\rm max}^2 = -2\gamma P_0/\beta_2$, and approaching zero at $\omega_0^2 = -4\gamma P_0/\beta_2$. In amplifiers, however, where the field power grows as $P(z) = P_0 e^{g_0 z}$, the most unstable frequency of perturbation increases during the propagation due to the power exponential growth. To estimate the growth due to MI in an amplifying medium, one can use the expression for the uniform MI but replace constant power with the growing one $P_0 \to P_0 e^{g_0 z}$. This corresponds to the so-called adiabatic approximation (see, e.g., [3]), in which it is assumed that the perturbation growth follows the intensity adiabatically and the standard MI expression with a z-dependent intensity $\gamma(z)$ can be used. Introducing $\eta = \frac{2\omega\sqrt{-\beta_2\gamma_0 P_0}}{g_0}$, $2s = g_0 T_2^2\omega^2$, $\mu = \frac{-\beta_2\omega^2}{g_0}$, and $p^2 = \frac{\mu^2}{\eta^2} = \frac{-\beta_2\omega^2}{4\gamma_0 P_0}$, the equations for $a(z)$ and $b(z)$ can be presented in the form

$$\frac{da}{dz} = \frac{g_0\mu}{2}b(z) - sa(z),$$
$$\frac{db}{dz} = \frac{\eta^2 g_0}{2\mu}(-p^2 + \exp[g_0 z])a(z) - sb(z). \qquad (2)$$

The solution to Eq. (2) can be presented through the Bessel functions $I_{i\mu}(x)$ and $K_{i\mu}(x)$ (compare to [10,11]):

$$a(z) = AI_{i\mu}(\eta e^{g_0 z/2}) + BK_{i\mu}(\eta e^{g_0 z/2}),$$
$$b(z) = -\frac{\eta e^{g_0 z/2}}{2\mu}\{C(I_{i\mu-1} + I_{-i\mu+1}) + D(K_{i\mu-1} + K_{i\mu+1})\}. \quad (3)$$

Here $A = [-a(0)\eta K'_{i\mu}(\eta) + b(0)\mu K_{i\mu}(\eta)]e^{-sz}$, $B = [a(0)\eta I'_{i\mu}(\eta) - b(0)\mu I_{i\mu}(\eta)]e^{-sz}$, $C = -\eta\mu^{-1}e^{g_0z/2}A$, and $D = \eta\mu^{-1}e^{g_0z/2}B$. The solutions of Eq. (3) are functions of three dimensionless parameters: $g_0 z$, μ, and η. The Stürmian theory [12] guarantees, for the Sturm–Liouville problem in Eq. (2) that the solutions in Eq. (3) are growing with z under condition $\mu < \eta e^{g_0z/2}$. For $\eta e^{g_0z/2} > \mu$ and $\mu \gg 1$, the leading term in the expansion of the exact solution reads

$$I_{i\mu}(\eta e^{g_0z/2}) \approx \frac{\exp\left[\sqrt{\eta^2 e^{g_0z} - \mu^2} + \mu \arcsin\left(\frac{\mu e^{-g_0z/2}}{\eta}\right)\right]}{\sqrt{2\pi}\sqrt[4]{(\eta^2 e^{g_0z} - \mu^2)}} + \ldots,$$

$$K_{i\mu}(\eta e^{g_0z/2}) \approx \frac{\sqrt{\pi}\times\exp\left[-\sqrt{\eta^2 e^{g_0z} - \mu^2} - \mu \arcsin\left(\frac{\mu e^{-g_0z/2}}{\eta}\right)\right]}{\sqrt[4]{4(\eta^2 e^{g_0z} - \mu^2)}} + \ldots. \quad (4)$$

It is seen that, in this limit, $K_{i\mu}$ is decaying and $I_{i\mu}$ is growing and the growth of perturbations in the amplifier is *superexponential*. In the opposite limit, $\mu > \eta e^{g_0z/2}$, both $K_{i\mu}$ and $I_{i\mu}$ are oscillating. Note that the asymptotic behavior of $I_{i\mu}$ not only justifies the use of the adiabatic approximation [3] in the limit $\mu \gg 1$, but also provides the preexponential factor. The increment of growth in the adiabatic approximation is $\Gamma = 2f(\mu, \eta, g_0, L) - 2sL$, with $f(\mu, \eta, g_0, L)$ defined as

$$f = (\eta^2 e^{g_0L} - \mu^2)^{1/2} - \sqrt{\eta^2 - \mu^2} + \mu \arcsin\left(\frac{\mu e^{-g_0z/2}}{\eta}\right) - \mu \arcsin\left(\frac{\mu}{\eta}\right), \quad \frac{\mu}{\eta} < 1,$$

$$f = (\eta^2 e^{g_0L} - \mu^2)^{1/2} + \mu \arcsin\left(\frac{\mu e^{-g_0L/2}}{\eta}\right), \quad \frac{\mu}{\eta} > 1.$$

The important result of our work is that it gives a direct analytical expression for the dynamics of the perturbations for *any arbitrary initial fluctuations and any propagation distance*. The power growth of the initial perturbations can be characterized by the increment factor (similar to the homogeneous case making comparison more convenient) defined as $\Gamma = 2\ln[a(L)/a(0)]$, $\mu < \eta$; $\Gamma = 2\times\ln[a(z)/a(z^*)]$, $\mu > \eta$, $z^* = \frac{1}{g_0}\ln\left[\frac{\mu^2}{\eta^2}\right]$. Here, we assume $a(0), a(z^*) \neq 0$. For large $a(L)/a(0)$, the increment is practically independent of boundary conditions. It should be stressed, however, that in the exact solutions of Eq. (4), there are both growing and decaying solutions. For short propagation distances, both can contribute to the development of instability—a fact that is often overlooked when considering MI. This means, in particular, that, for short devices where MI does not have enough time/distance to develop into an asymptotic state with the growing mode dominating completely, the initial phase perturbations given by $b(0)$ might affect the growth increment of developing modulations. Initial conditions also become important near the cutoff of instability, as the growth is not large near such points and it is influenced by the initial field perturbations. This is illustrated by Fig. 1, where the relative impact of the initial phase

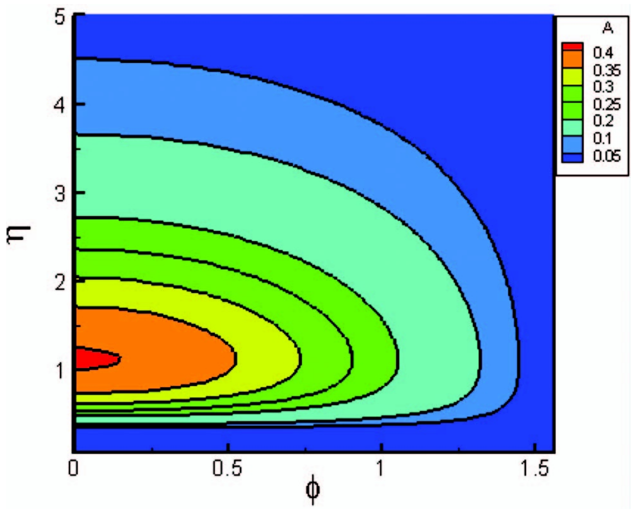

Fig. 1. Counterplot of the coefficient $A = [-a(0)\eta K'_{i\mu}(\eta) + b(0)\mu K_{i\mu}(\eta)]$ before the growing solution in the plane (η, ϕ) with $\mu = 1$.

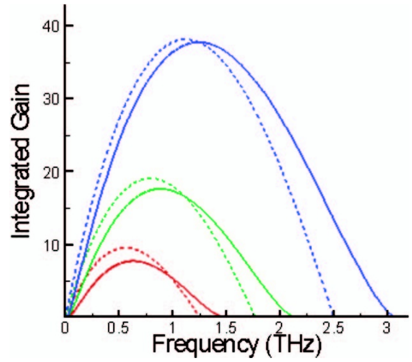

Fig. 2. Gain $\Gamma(\nu)$ for $L = 100$ m, $G = 30$ dB. Here $P_0 = |\Psi_0|^2$: 50 (lower red curves), 100 (central green curves) and 200 mW (top blue curves). Solid curves, exact solutions; dashed curves, adiabatic approximation [3].

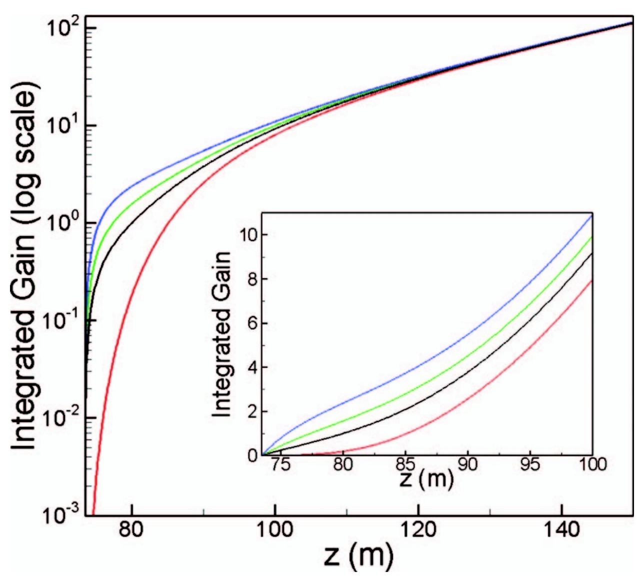

Fig. 3. Integrated gain Γ (log scale) versus propagation distance for $a(z^*) = 1$ and $b(z^*) = 0$ (red curve), $b(z^*) = 1$ (green curve), $b(z^*) = 2$ (blue curve); black curve, $I_{i\mu}$, $P_0 = 50$ mW. Inset, normal scale.

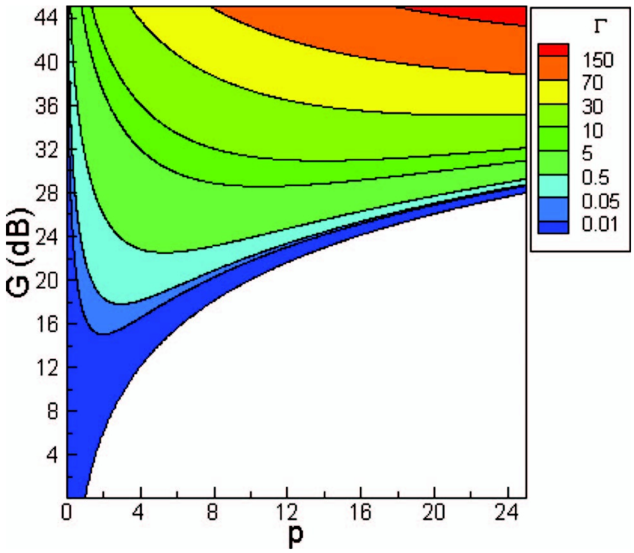

Fig. 4. Counterplot Γ in the plane (p,G), $\eta^2/\mu = 0.03$. The white zone corresponds to the oscillating solutions. The border between stable and unstable regimes is given by the condition $G > \mu^2/\eta^2$.

$b(0)$ and amplitude $a(0)$ perturbations on the growing solution are shown (the coefficient before the growing solution $A = [-a(0)\eta K'_{i\mu}(\eta) + b(0)\mu K_{i\mu}(\eta)]$). Here $a^2(0) + b^2(0) = 1$ and $\phi = \tan^{-1}[b(0)/a(0)]$. In general, the increment factor $\Gamma(\mu,p,s,G,L)$ is a multiparametric function of the parameters μ, p, s, G, and L or, in the real-world units, ω, P_0, β_2, γ, T_2, G, and L. Therefore, the existence of the analytical solution is very useful for design analysis. For fixed values of other parameters, we have to determine the maximum value of the increment growth Γ as a function of ω. In a uniform media ($g_0L = 0$), the most unstable mode corresponds to $p^2 = 1/2$ and cutoff at $p^2 = 1$. In contrast, in an amplifier, the most unstable value of ω increases during the propagation. For illustration, we use here similar parameters as in [3]: $\beta_2 = -20$ ps^2/km, $\gamma = 10$ W^{-1} km^{-1}, amplifier length $L = 100$ m, and total gain $G = g_0L = 30$ dB. Figure 2 shows the integrated gain $\Gamma(\nu)$ for several values of the input power. It is seen that the adiabatic approximation (dashed curves), being close to the exact solutions (solid curves), still deviates in determination of the frequency of the maximal instability. This might be critically important for design of MI-based lasers. Figure 3 illustrates the impact of the initial conditions on the instability growth (typically overlooked in studies limited by the analysis of the growth increment only), showing growth of the integrated gain Γ with distance for $a(z^*) = 1$ and different $b(z^*)$: $b(z^*) = 0$, red curve; $b(z^*) = 1$, green curve; $b(z^*) = 2$, blue curve. Here, the black curve corresponds to $I_{i\mu}$. Figure 4 depicts the integrated gain x as a function of the normalized frequency p and the total gain G. Note that the p corresponding to maximum MI growth shifts up with g_0L increasing; the cutoff takes place at $p > 1$, and the most unstable modes correspond to $p > 1$. This means that the most unstable modes initially were stable and start to grow only later downstream.

We have revisited the theory of MI fiber amplifiers. We derived the complete analytical solutions of the linear growth that allows us to find the most unstable mode and calculated the power growth exactly—without restricting the consideration to the asymptotically growing mode, as in most previous works. We demonstrated that, for practical situations, the growth of the perturbation is sensitive to the initial perturbation and to their phases. Our results are directly relevant to the MI in optical fiber amplifiers and lasers, but the derived theory is rather general and can be applied in a variety of physical applications beyond fiber optics.

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