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Stability of an optical soliton with Gaussian tails

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The structure and stability of a soliton with Gaussian tails is studied in the model that presents the nonlinear Schrödinger equation with additional parabolic potential. It is suggested to use in-line phase modulators to create and to transmit such pulses in cascaded fiber links. It is proved that the soliton with Gaussian tails is stable and can be potentially used as an information carrier in the optical transmission.
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Interaction between two neighboring pulses is one of the main factors limiting bit rate achievable by modern soliton-based transmission systems [1–3]. Overlap of the exponential tails of the closely spaced pulses leads to perturbative interaction of the solitons and the system performance degradation. To provide stable transmission, a separation between two neighboring fundamental solitons typically should be not less than five soliton widths. This is a principal limitation for transmission based on soliton with sech shape described by the nonlinear Schrödinger equation (NLSE).

One possible way to increase transmission bit rate is to use as an information carrier a solitary wave with the tails decaying faster than exponential wings of the NLSE soliton. This results in a substantial suppression of the soliton interaction and consequently in a possibility of a more dense information packing. Stable propagation of the linear Gaussian pulses cannot be realized in a conventional fiber line, because of the nonlinear Kerr effect that in a combination with dispersion leads to the generation of the solitons with sech-type profiles. The Gaussian tails can result, however, from an effective pulse chirping combined with strong dispersion management or other physical mechanisms such as, for instance, phase modulation. This kind of soliton has been discussed in recent works [4–6]. In [6], for instance, it has been shown that a stationary quasisoliton can be formed using specially programmed chirp and dispersion profile. To avoid misleading with using the term soliton, note that in contrast to the NLSE soliton, a solitary pulse with Gaussian tails is not a soliton in terms of the integrable models. However, from the viewpoint of practical applications this difference is not critical. Numerical simulations show that such a pulse holds most of the important features of the fundamental soliton. Stability is the central issue for the physical relevance of the localized pulses with Gaussian tails and their implementation as information carriers in the high-bit-rate optical transmission systems. Long-term ''averaged'' dynamics of the soliton with Gaussian tails in some systems of practical importance is governed by the NLSE with additional parabolic potential. As an example, we point out in this Rapid Communication that the use of in-line phase modulators in the fiber transmission lines allows us to create an effective parabolic potential for propagating the soliton. This leads to the formation of nonlinear carrier pulses with Gaussian tails. The steady-state solitary wave solution in such a model presents an intermediate state between the NLSE sech-type soliton and a Gaussian pulse. In this Rapid Communication the stability of the soliton with Gaussian tails is proved. We demonstrate that the solitary wave solution in the considered model has fast decaying tails in comparison with the sech-type soliton. Note also that the considered model is rather general and occurs in different physical contexts, for instance, it describes a pulse propagation in the preformed plasma channel (see, e.g., [7,8] and references therein).

Consider as a basic model for average evolution of the soliton with Gaussian wings the following dimensionless modification of the NLSE (see, e.g., [5,6]):

$$iQ_z+\frac{1}{2}Q_{tt}+|Q|^2Q-at^2Q=0\,. \tag{1}$$

Here, depending on the specific problem, t is either a normalized time, or the self-similar variable related to time [5,6,9]. Parameter a is assumed to be positive, $a>0$. This equation includes the most important effects, namely, dispersion (or averaged dispersion), nonlinearity, and the effective parabolic potential that can result from different physical mechanisms. Note that such an equation has been derived recently in the context of optical pulse dynamics in the links with dispersion compensation and quasisoliton propagation along the line with special dispersion profile [5,6,9]. In the case of a dispersion-managed soliton with maps considered in [4,5,9] an effective potential is of a nontrapping type ($a<0$). In [10] it has been shown, however, that additional grating can provide Gaussian tails of the carrier pulse. Insertion of in-line phase modulators in the cascaded optical communication line can be also described in an average by Eq. (1) with positive a. Similar models occur in the description of the mode-locking laser systems using electro-optical phase modulators. Because this equation seems to be a fundamental generic model describing propagation of a soliton with Gaussian tails in different practical realizations, in this Rapid Communication we concentrate mostly on the general properties of the model rather than on specific applications. We consider here only the trapping potential with $a>0$.

Equation (1) can be written in the Hamiltonian form

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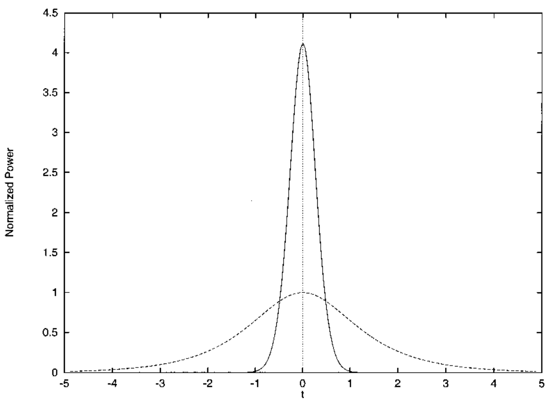

FIG. 1. Shape of the soliton with Gaussian tails. It is plotted power distribution $|F(t)|^2$ (normalized to the peak power of the fundamental soliton) in the solutions of Eq. (1) with $k=0.5$ for different values of a vs normalized time t. Dashed line, soliton of the NLSE ($a=0$); solid line, the soliton with Gaussian wings for $a=10$.

$$i\frac{\partial Q}{\partial z}=\frac{\delta H}{\delta Q^*}, \tag{2}$$

with the Hamiltonian

$$H=\frac{1}{2}\int |Q_t|^2 dt-\frac{1}{2}\int |Q|^4 dt+a\int t^2|Q|^2 dt=I_1-I_2+I_3. \tag{3}$$

The integral $E=\int|Q|^2dt$ is an additional conserved quantity typically having the meaning of the energy.

Consider a steady-state soliton solution of the form

$$Q(z,t)=F(t)\exp[ikz]; \tag{4}$$

The soliton shape is given by the equation

$$-kF+\frac{1}{2}F_{tt}+|F|^2F-at^2F=0. \tag{5}$$

One can see that a shape of the localized pulse is an intermediate state between the sech-type profile of the NLSE soliton and the Gaussian pulse. The typical pulse profile for different a is shown in Fig. 1. In what follows we consider a ground state that is real (apart from a constant phase factor). Note that the parameter k can be negative for some localized solutions. In the limit $a=0$ the solution is a soliton of the NLSE $F=\sqrt{2k}/\cosh(\sqrt{2k}t)$. In the opposite limiting case $a\rightarrow\infty$, the solution is close to a Gaussian pulse $F=\exp[-\sqrt{a}t^2/\sqrt{2}]$ and the latter describes also asymptotic decreasing of the F for large t. It is clear that the tails of the localized pulse with $a\neq 0$ decay much faster than the exponential wings of the fundamental soliton ($a=0$). As a result, solitons with Gaussian tails can be spaced much closer to each other still keeping interaction between neighboring pulses suppressed [6,11]. Obviously, this allows more dense packing of the information. This bright-type soliton exists in the anomalous dispersion region (in application to the dispersion compensating systems, the residual dispersion must be anomalous). In Fig. 2 is plotted the energy of the soliton E as a function of the parameter a for $k=0.5$. An important feature of the soliton with Gaussian tails is the enhancement of the energy in comparison with the NLSE soliton for the same average dispersion ($a=0$). However, comparing solitons with the same pulse widths it can be shown that the energy of the considered soliton with Gaussian tails is less than the energy of a corresponding sech-type soliton. The enhancement of the energy of the dispersion-manged soliton observed in [4] is due to the tunneling of the radiation in the case $a<0$. For the dispersion compensating systems [5] the parameter a depends on the characteristics of the dispersion map, pulse power, and residual dispersion. We do not consider in this publication the case of negative a that typically takes place for dispersion-managed soliton.

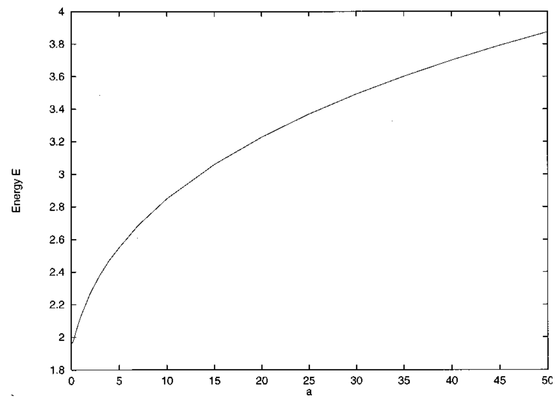

FIG. 2. Enhancement of the energy E with increasing a. The energy of the soliton is plotted as a function of the parameter a for $k=0.5$. The energy of the fundamental soliton $E=2$ corresponds to $a=0$.

Equation (5) can be presented in a variational form

$$\delta(H+kE)=0; \tag{6}$$

from here it is easy to find that the soliton solution realizes the extremum of H for a fixed E. This variational form can be used to gain some qualitative information about stability. Consider scaling transformation of the Hamiltonian H keeping E constant: $Q=F(t/\alpha)/\sqrt{\alpha}$. Using such a trial function we get for H

$$H(a)=\frac{I_1}{\alpha^2}-\frac{I_2}{\alpha}+\alpha^2 I_3. \tag{7}$$

The first two terms are associated with the NLS equation. It is clear from Eq. (7) that H has a global minimum. Using well-known properties of the Sturm-Liouville operators, it is easy to show that this global maximum is attained at the solution without zeroes [$(F(t)>0$].

To study the stability of a solitary pulse [Eq. (5)] let us consider the evolution of small perturbations on the soliton solution. Let us linearize Eq. (1) on the soliton and decompose the perturbation into real and imaginary parts $Q=(F+f+ig)\exp[ikz]$. After standard consideration, making Fourier transformation we obtain the following spectral problem:

$$\omega^2 f=H_+H_-f; \tag{8}$$

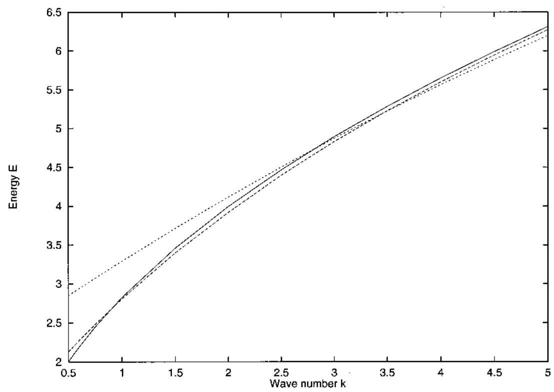

FIG. 3. The energy of the soliton as a function of the wave number k for different a: solid line for $a=0$ (fundamental soliton) dashed line close to solid one is for $a=1$, and dotted curve for $a=10$. Monotonic growth of the energy with increasing of k determines the stability of the soliton with Gaussian wings.

here

$$H_+ = k - \frac{1}{2}\frac{d^2}{dt^2} - |F|^2 + at^2, \quad H_- = H_+ - 2|F|^2. \qquad (9)$$

The stability of a solitonic pulse is determined by the properties of the operators H_+, H_-. It is easy to check that $H_+F=0$, and $H_-F_t=-2atF$. Additionally $H_+tF=-F_t$ and $H_-(\partial F/\partial k)=-F$.

Operator H_+ is nonnegative, because the eigenfunction F has no zeros and, consequently, corresponds to the lowest eigenvalue. The minimum of ω^2 can be found as

$$\omega^2 = \min \frac{\langle f|H_- f\rangle}{\langle f|H_+^{-1} f\rangle}; \qquad (10)$$

here $\langle f|g\rangle \equiv \int f^* g dt$ and the minimum is considered in the subspace of functions orthogonal to F. From the latter expression it is seen that the stability is determined by the existence of a negative eigenvalue of the operator H_- with an additional orthogonality condition $\langle F|f\rangle=0$. Using the standard technique (see, e.g., [12,13] for details), it is easy to find that the stability of the soliton is determined by a sign of the first derivative of the energy E with respect to k. Omitting mathematical details, a sketch of the proof is as follows. Presenting f in the form $f=\partial F/\partial k^2 - Cg$ with C $\langle F|g\rangle = \langle F|\, \partial F/\partial k^2\rangle$ we reformulate problem of the minimization of the functional $\langle f|H_- f\rangle$ under additional constraint $\langle F|f\rangle=0$ as a problem of a determination of the absolute minimum of the functional $G[g] = \langle g|H_- g\rangle/(\langle F|g\rangle^2)$.

It can be found that under condition

$$\frac{\partial E}{\partial k} > 0 \qquad (11)$$

this absolute minimum of $G[g]$ is attained and the corresponding minimum of $\langle f|H_- f\rangle$ (under condition $\langle F|f\rangle=0$) is zero. Thus, under this condition, there is no growing mode and the condition (11) is the criterion of the soliton stability. In Fig. 3 E versus k for different fixed a is plotted. Energy E monotonically grows with increasing of the wave number k, therefore, the solitons with Gaussian tails described by Eq. (1) are stable.

One can find some integral condition on the characteristics of an input signal which provides that a maximum of the peak power will be bounded from below by a constant determined by the input signal parameters. First, let us estimate the Hamiltonian H using the following inequality, $\sqrt{a}E = -\sqrt{a}\int t(QQ_t^* + Q^*Q_t)dt \leqslant \int |Q_t|^2 dt + a\int t^2|Q|^2 dt$:

$$2H = \int |Q_t|^2 dt - \int |Q|^4 dt + 2a\int t^2|Q|^2 dt$$
$$\geqslant \sqrt{2a}E - \max|Q|^2 E. \qquad (12)$$

Because H and E are conserved quantity, we can estimate the maximum of a pulse peak power at any z from below:

$$\max|Q|^2 \geqslant \sqrt{2a} - \frac{2H}{E}. \qquad (13)$$

Thus, if an input pulse satisfies the condition $2H \leqslant \sqrt{2a}E$, then a maximal peak power cannot decrease with pulse evolution below some constant value that is estimated by the right-hand-side of Eq. (13). This indicates that the energy cannot be dispersed among linear modes in a way that a peak power decreases below some constant value.

Additional information about pulse dynamics in the system described by Eq. (1) can be obtained considering evolution of the average square of the pulse width. Define R as $R = \int t^2|A|^2 dt / \int |A|^2 dt$. This quantity has a meaning of the average square of a pulse width,

$$\frac{d^2R}{dz^2} = 2\frac{H}{E} + 2\frac{I_1}{E} - 6aR \geqslant 2\frac{H}{E} + \frac{1}{2R} - 6aR = -\frac{\partial W(R)}{\partial R}; \qquad (14)$$

the latter representation allows us to use the analogy with the motion of a particle in the effective potential $W(R) = 3aR^2 - 2HR/E - 0.5\ln(R)$ treating z as the effective ''time.'' One observation that can be seen from this analogy is that an average pulse width can reach neither zero nor infinity. Of course, this does not prohibit compression of the central peak on the broadening background.

In conclusion, the stability of the soliton with Gaussian tails is proved. Such a soliton is an intermediate state between the NLSE sech-type soliton and a Gaussian pulse. Fast decay of the Gaussian tails leads to a substantial reduction of the soliton interaction and consequently to a possibility of much dense information packing in comparison with the fundamental solitons. Stability of the solitons with Gaussian tails makes them promising candidates for use as information carriers in high-bit-rate transmission systems. It is suggested to use in-line phase modulators to transmit solitons with Gaussian tails.

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