Stability of magnetoelastic solitons and self-focusing of sound in antiferromagnet

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Truncated equations describing the evolution of sound waves in antiferromagnets are derived. The crystallographic anisotropy is taken into account. These equations are used to study the stability of plane solitons. A theorem on self-focusing is proved. Classes of initial data which give rise to a self-focusing sound wave are identified.

INTRODUCTION

The propagation of elastic waves in antiferromagnets has several remarkable aspects, which have recently attracted interest.[1–4] The reason for these remarkable features is a strong magnetostrictive interaction,[1] which gives rise to a significant nonlinearity of the elastic subsystem of the crystal.[2] In most solids, at the strain levels which are achievable in practice, nonlinear acoustic effects are weak and difficult to observe.[5] In antiferromagnets, in contrast, the coupling with the magnetic subsystem causes the nonlinearity and the dispersion of sound waves to be far greater than in the purely acoustic case. The result is an unusual possibility to study the dynamics of nonlinear and dispersive sound waves in solid-state experiments, in which highly accurate measurements are possible.

Several extremely simple nonlinear effects which arise during the propagation of sound in antiferromagnets have been studied experimentally and theoretically in recent years: second-harmonic generation,[3] self-effects,[6] nonresonant interactions,[7] and the stimulated Raman scattering [4] of sound waves. It is now becoming possible to study effects which correspond to higher-order nonlinearities, primarily the formation of solitons and the self-focusing of sound (which have previously been seen only in numerical simulations[8]). Estimates[9,10] show that it would be feasible to observe these effects even with the experimental facilities available today. A systematic search for solitons and self-focusing requires a preliminary study of two questions: Under which conditions are solitons stable? From which intial distributions does a self-focusing sound wave form? Our purpose in the present paper is to answer these questions.

At a strain level which is not too high, the evolution of an elastic wave can be separated into fast and slow components. The fast component is the transport of the initial perturbation at the velocity of sound, while the slow component is caused by weak nonlinearity and dispersive effects. To study these effects it is convenient to use a coordinate system which is moving at the sound velocity, retaining in the equations the terms which are the most important for the nonlinearity and the dispersion. This approach corresponds to the standard procedure for constructing truncated equations.[11] In §1 we derive these equations, which describe the evolution of sound waves, with allowance for the crystallographic structure of the antiferromagnets used experimentally: rhombohedral (α-Fe_2O_3, Fe_2BO_3, $MnCO_3$) and orthorhombic ($TmFeO_3$, etc.). It turns out that, despite the many different cases which are possible (for various directions of the external magnetic field, of the propagation, and of the sound polarization), the one-dimensional dynamics can be described by two universal equations. Specifically, when there is a linear magnetoelastic coupling (in other words, when the sound velocity depends on the magnetic field) the slow evolution of the strain tensor u obeys the equation

$$u_t+(n+2)(n+1)u^n u_x+Du_{xxx}=0.$$

In the case of a common position we would have $n=1$, but for several wave propagation directions the coefficient in the term with the quadratic nonlinearity vanishes, and we have $n=2$. The sign of the dispersion is determined by the relation between the linear volocities of magnons (V_m) and phonons (V_s) (the magnetoelastic renormalization is taken into account): $\operatorname{sign} D=\operatorname{sign}(V_s-V_m)$. The weakly three-dimensional dynamics can be described by the equation

$$\frac{\partial}{\partial x}[u_t+(n+2)(n+1)u^n u_x+Du_{xxx}]=au_{yy}+cu_{zz}. \qquad (1)$$

Curiously, the signs of a and c may differ in certain cases (§1).

For sound waves, which are not coupled with magnons in the linear approximation, we find the system of equations

$$u_t+\varphi\varphi_x=0, \quad -D\varphi_{xx}=\varphi+u\varphi. \qquad (2)$$

Here φ is the angle through which the antiferromagnetism vector is rotated from its equilibrium position.

The properties of Eqs. (1) and (2) are studied in §2. In case (1), with $n=1$, we find the Kadomtsev-Petviashvili equation, which has been studied in detail.[8,11–14] In this equation there are plane solitons for either sign of D. With $D>0$, $a>0$, and $c>0$, both the solitons and periodic waves are unstable with respect to a transverse rippling,[12,13] and self-focusing is possible in this case, as numerical calculations have shown.[8] With $D<0$, $a>0$, and $c>0$, solitons are stable. In the case $n=2$, soliton solutions exist only if $D>0$; if $a<0$ and $c<0$, they are stable, but if the sign of either (or both) of a and c is positive the solitons are unstable with respect to rippling. With $a>0$ and $c>0$, as is proved in §2, there can be a self-focusing for initial conditions which satisfy $\mathscr{H}<0$, where

$$\mathscr{H}=\int\left[\frac{D}{2}u_x^2+\frac{a}{2}w_y^2+\frac{c}{2}w_z^2-u^n\right]d\mathbf{r}, \qquad w_x=u.$$

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In system (2), solitons exist only if $D < 0$. The absence of soliton solutions [in (1) with $n = 2$, $D < 0$, and in (2) with $D > 0$] apparently implies that the initial perturbations spread out, since the nonlinearity and the dispersion "act in the same directions" in this case. Even if there are soliton solutions (with $D < 0$), however, they are unstable even in a one-dimensional system, (2). It might be hypothesized that the result of the onset of this instability in system (2) is the formation of a singularity in a finite time, i.e., a collapse.

In the Conclusion we summarize the results regarding the possibility of observing solitons and self-focusing in some specific ferromagnets: α-Fe_2O_3, $MnCO_3$, and $TmFeO_3$.

§1. DERIVATION OF TRUNCATED EQUATIONS

1.1 *Crystals of rhombohedral symmetry.* In this subsection we consider an antiferromagnet with an easy-plane anisotropy. The sublattice magnetic moments $\mathbf{M}_1$ and $\mathbf{M}_2$ ($|\mathbf{M}_1| = |\mathbf{M}_2| = M_0$) lie in the basis plane (the xy plane). It is convenient to transform to the variables $\mathbf{m} = (\mathbf{M}_1 + \mathbf{M}_2)/2M_0$, $\mathbf{l} = (\mathbf{M}_1 - \mathbf{M}_2)/2M_0$; then we obviously have $m^2 + l^2 = 1$, $(\mathbf{ml}) = 0$. The free-energy density of the magnetic subsystem of the sample is[2]

$$F_m=2M_0[H_E m^2-H_D[\mathbf{ml}]_z+{}^1/_2H_A l_z^2+{}^1/_2\alpha M_0(\nabla\mathbf{l})^2-(\mathbf{mH})]. \tag{3a}$$

The energy is minimized in the state with $m \neq 0$, $l_z = 0$, $\mathbf{m}\|\mathbf{H}$, $\mathbf{m}\perp\mathbf{l}$. We write $\mathbf{H} = H(\cos\varphi_0, \sin\varphi_0, 0)$; then in our analysis of magnetoelastic oscillations we will assume $\mathbf{l} = (-\sin\psi, \cos\psi, 0)$, $\psi = \varphi_0 + \varphi$, $\varphi \ll 1$. The energy of the elastic subsystem of the crystal is expressed in terms of the components of the strain tensor u_{ij}:

$$\begin{aligned}F_e={}&{}^1/_2C_{11}(u_{xx}^2+u_{yy}^2)+{}^1/_2C_{33}u_{zz}^2+C_{12}u_{xx}u_{yy}\\&+C_{13}(u_{xx}+u_{yy})u_{zz}+(C_{11}-C_{12})u_{xy}^2\\&+2C_{44}(u_{xz}^2+u_{yz}^2)+2C_{14}[(u_{xx}-u_{yy})u_{yz}+2u_{xy}u_{xz}].\end{aligned} \tag{3b}$$

In writing the magnetoelastic energy density we consider only the terms which are nonlinear in u_{ij} (as in Ref. 2), assuming that the strain is small:

$$\begin{aligned}F_{me}={}&B_{11}(l_x^2u_{xx}+l_y^2u_{yy})+B_{12}(l_x^2u_{yy}+l_y^2u_{xx})\\&+2(B_{11}-B_{12})l_xl_yu_{xy}+B_{33}l_z^2u_{zz}+2B_{44}(l_yl_zu_{yz}\\&+l_xl_zu_{xz})+2B_{14}[2l_xl_yu_{xz}+(l_x^2-l_y^2)u_{yz}]\\&+B_{41}[l_yl_z(u_{xx}-u_{yy})+2l_xl_zu_{xy}].\end{aligned} \tag{3c}$$

In (3), H_E is the exchange field; H_A is the anisotropy field; H_D is the Dzyaloshinskiĭ field; α is a constant of the nonuniform exchange interaction; H is the external field, which lies in the basis plane; ρ is the crystal density; the x axis runs along one of the twofold axes U_2; and C_{ij} and B_{ij} are the constants of the elastic and magnetoelastic interactions. To find some numerical estimates of these quantities, we make use of experiments with hematite, with the parameter values[2] $C_{11} = 24.2$ (the Cs are given in units of 10^{11} erg/cm^3), $C_{33} = 22.6$, $C_{12} = 5.5$, $C_{44} = 8.5$, $C_{13} = 1.6$, and $C_{14} = -1.3$; $\rho = 5.29$ g/cm^3; $B_{11} - B_{12} = 8$ (B_{ij} is given in units of 10^6 erg/cm^3), $2B_{14} = 27$, and $2B_{44} = 53$; $M_0 = 870$ Oe; $H_E = 9.2\cdot 10^6$ Oe; and $H_D = 2.2\cdot 10^4$ Oe.

We first consider waves which are propagating along the x axis. If the initial strain field is uniform in the yz plane, then the only nonvanishing components of the strain tensor are u_{xx}, u_{xy}, and u_{xz}. The oscillations of the vector $\mathbf{l}$ are described by the equation (see Ref. 10, for example)

$$\gamma^{-2}[\mathbf{l},\Box_m\mathbf{l}]_z=(\mathbf{Hl})\{H_D+[\mathbf{H},\mathbf{l}]_z\}+2H_E[\mathbf{l},H_{me}]_z,$$

$$\Box_m=\frac{\partial^2}{\partial t^2}-V_m^2\frac{\partial^2}{\partial x^2},\qquad V_m^2=2H_EM_0\alpha,$$

$$\mathbf{H}_{me}=-\frac{1}{2M_0}\frac{\partial F_{me}}{\partial \mathbf{l}},$$

where H_{me} is the effective magnetoelastic field, and γ is the gyromagnetic ratio. Calculating H_{me}, and transforming from $\mathbf{l}$ to φ, we find the equation

$$\begin{aligned}\gamma^{-2}\Box_m\varphi={}&-H\sin\varphi\,(H\cos\varphi+H_D)\\&-H_E\frac{B_{11}-B_{12}}{M_0}\sin 2(\varphi_0+\varphi)u_{xx}\\&+\frac{2H_E(B_{11}-B_{12})}{M_0}\cos 2(\varphi_0+\varphi)u_{xy}+\frac{4H_EB_{14}}{M_0}\cos 2(\varphi_0+\varphi)u_{xz}.\end{aligned} \tag{4}$$

Assuming $H_A \gg H_{me}$ we have $l_z = 0$ everywhere.

Sound waves are described by the equations of elastic theory:

$$\rho\ddot{u}_i=\frac{\partial}{\partial x_i}\frac{\partial F}{\partial u_{ij}},\qquad u_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\right), \tag{5}$$

$$\left(\frac{\partial^2}{\partial t^2}-v_1^2\frac{\partial^2}{\partial x^2}\right)u_{xx}=\frac{\partial^2}{\partial x^2}\left[B_{11}\frac{1}{\rho}\sin^2(\varphi_0+\varphi)+\frac{B_{12}}{\rho}\cos^2(\varphi_0+\varphi)\right], \tag{6a}$$

$$\left(\frac{\partial^2}{\partial t^2}-v_6^2\frac{\partial^2}{\partial x^2}\right)u_{xy}=\frac{\partial^2}{\partial x^2}\left[2v_{14}^2u_{xz}-\frac{b}{2\rho}\sin 2(\varphi_0+\varphi)\right], \tag{6b}$$

$$\left(\frac{\partial^2}{\partial t^2}-v_4^2\frac{\partial^2}{\partial x^2}\right)u_{xz}=\frac{\partial^2}{\partial x^2}\left[2v_{14}^2u_{xy}-\frac{B_{14}}{\rho}\sin 2(\varphi_0+\varphi)\right]. \tag{6c}$$

Here

$$v_1^2=C_{11}/\rho,\quad v_6^2=(C_{11}-C_{12})/\rho,\quad v_4^2=2C_{44}/\rho,\quad v_{14}^2=C_{14}/\rho,\quad b=B_{11}-B_{12}.$$

The directions $\varphi_0 = \pi n/4$, $n = 0, 1, 2, \ldots$, are special directions for the field H, as can be seen from (6). For these values of φ_0, not all of the strain components are coupled with magnetic oscillations in the linear approximation, and there is no quadratic nonlinearity in the equations for this case.

Let us assume, for example, $\varphi_0 = \pi/4$. In this case, three quasielastic modes can propagate. One of them (a longitudinal mode) is coupled with magnetic oscillations even in the linear approximation. We can derive an equation describing the slow evolution of $\varphi(x,t)$ in a coordinate system which is moving at the longitudinal sound velocity. Assuming that the dispersion and the nonlinearity are slight, we express u_{xx} in terms of φ from (4), and we substitute the result into (6a). We then transform to the moving coordinate system, $\partial/\partial t \to \partial/\partial t - v_{1H}\partial/\partial x$, assuming $\varphi_t \ll v_{1H}\varphi_x$, where

$$v_{1H}^2=v_1^2-v_H^2=v_1^2-H_Eb^2/\rho M_0H(H+H_D).$$

We find a modified Korteweg–de Vries equation

$$\varphi_t+\frac{v_{1H}^2-v_m^2}{\Omega^2}\varphi_{xxx}+\frac{7}{2}\frac{v_H^2}{v_m^2}\varphi^2\varphi_x=0.$$

Here $\Omega=\gamma[H(H+H_D)]^{1/2}$ is the antiferromagnetic resonance frequency.

In addition to the longitudinal sound there are two transverse modes, whose velocities (v_+ and v_-) are independent of the field in the linear approximation for $\varphi_0=\pi/4$:

$$v_\pm^2=\frac{v_4^2+v_6^2}{2}\pm\left[\left(\frac{v_4^2-v_6^2}{2}\right)+4v_{14}^4\right]^{1/2}.$$

Again using the approximation in which the modified Korteweg–de Vries equation is found, and transforming to a coordinate system which is moving at v_+, we find the system of equations

$$\frac{v_+^2-v_m^2}{\gamma^2}\varphi_{xx}=-H(H+H_D)\varphi-\frac{2H_Eb}{M_0}\varphi W,$$

$$W_t=-\frac{1}{2v_+(v_+^2-v_-^2)^2\rho}\left[4B_{14}v_{14}^2+\frac{b}{2}(v_1^2-v_4^2)+\frac{2B_{14}^2}{b}(v_1^2-v_6^2)\right]\frac{\partial\varphi^2}{\partial x},$$

$$w=u_{xy}+\frac{2B_{14}}{b}u_{xz}.$$

Here, under the assumption $\partial/\partial t\ll v_+\partial/\partial x$, we are ignoring the time derivatives and the terms φ^3 in comparison with φ in the first equation; in the second equation, since the terms with $\partial^2/\partial x^2$ cancel out, we are retaining the term $\partial^2/\partial x\partial t$, ignoring the higher-order time derivatives. A system of equations describing the slow evolution of waves propagating at a velocity near v_- is found by interchanging v_+ and V_-.

Any real strain field which is produced will be nonuniform in the plane perpendicular to the propagation line. To derive an equation which describes the slow evolution of slightly nonuniform perturbations, we should incorporate in the equations the other components u_{yy}, u_{yz}, and u_{zz} of the strain tensor.

Assuming that the motion is weakly three-dimensional ($\partial/\partial x\gg\partial/\partial y,\partial/\partial z$), we restrict the analysis to the linear approximation in the terms which contain derivatives with respect to the transverse coordinates, and as a result we find a modified Kadomtsev-Petviashvili equation in place of the modified Korteweg–de Vries equation for perturbations which are propagating at $v=v_{1H}$:

$$\partial/\partial x(\varphi_t+D\varphi_{xxx}+Q\varphi^2\varphi_x)=a_{11}\varphi_{yy}+a_{12}\varphi_{yz}+a_{22}\varphi_{zz},\qquad(7)$$

where

$$-2v_{1H}a_{11}=\frac{2v_{14}^4}{v^2}+\frac{2v_6^2}{v^2-v_6^2}\left(1+\frac{v_{12}^2}{v^2}\right)(v_1^2+v_{12}^2)+\frac{2v_{14}^4}{v^2-v_4^2}\left(2+\frac{v_4^2}{v^2}+\frac{2v_{12}^2}{v^2}\right)\left(1+\frac{v_1-v_{12}^2}{v^2-v_6^2}\right),$$

$$-2v_{1H}a_{22}=\frac{2v_{14}^4+v_{13}^4}{v^2}+\frac{4v_{14}^4}{v^2-v_6^2}\left(1+\frac{v_6^2}{2v^2}+\frac{v_{13}^2}{v^2}\right)\left(2+\frac{v_1^2+v_{13}^2}{v^2-v_4^2}\right)+\frac{v_4^2}{v^2-v_4^2}\left(1+\frac{v_{13}^2}{v^2}\right)\left(v_1^2+v_{13}^2+\frac{4v_{14}^4}{v^2-v_6^2}\right),$$

$$-a_{12}=v_{1H}\left[\frac{v_{12}^2+v_{13}^2}{v^2}+\frac{v_6^2}{v^2-v_6^2}\left(1+\frac{v_{12}^2}{v^2}\right)\left(2+\frac{v_1^2+v_{13}^2}{v^2-v_4^2}\right)+\frac{v^2}{v^2-v_4^2}\left(1+\frac{v_{13}^2}{v^2}\right)\left(1+\frac{v_1^2+v_{12}^2}{v^2-v_6^2}\right)+\frac{1}{v^2-v_4^2}\left(\frac{2v_{14}^2}{v^2-v_6^2}+\frac{v_1^2+v_{13}^2}{2}\right)\left(2+\frac{v_4^2}{v^2}+\frac{2v_{12}^2}{v^2}\right)+2\frac{v_1^2+v_{12}^2}{v^2-v_6^2}\left(1+\frac{v_6^2}{2v^2}+\frac{v_{13}^2}{v^2}\right)\right].$$

In writing a_{11}, a_{12}, and a_{22} we have discarded terms which are small, on the order of the parameter $H_Eb^2/\rho M_0H(H+H_D)=v_H^2/v_1^2$ (this parameter is of order 10^{-2} for hematite with $H\approx1$ kOe), and we have used the condition $C_{14}\ll C_{11}$ to simplify the resulting expressions. The replacement

$$y=\alpha y'+\beta z',\quad z=\alpha z'-\beta y',\qquad 2\alpha^2=1-\left[1-\frac{a_{12}^2}{a_{12}^2+(a_{11}-a_{22})^2}\right]^{1/2},\quad\beta^2=1-\alpha^2,\qquad(8)$$

can be used to diagonalize the right side of (7), which becomes $-a'\varphi_{z'z'}-c'\varphi_{y'y'}$. For hematite with $H=1$ kOe we find $a'\approx3.5\cdot10^5$ cm/s and $c'\approx2.8\cdot10^5$ cm/s.

The absence of first derivatives with respect to the transverse coordinates in (7) is a consequence of the degeneracy of this particular case. There would of course be first derivatives with respect to the transverse coordinates for arbitrary directions of the field and the wave velocity. For example, in the case

$$\mathbf{v}\|y,\quad v^2=v_{6H}^2=v_6^2-\frac{2H_Eb^2}{\rho M_0H(H+H_D)},\quad\varphi_0=0$$

we would replace (7) by the equation

$$\frac{\partial}{\partial y}(\varphi_t+D_1\varphi_{yyy}+Q_1\varphi^2\varphi_y)=\alpha\varphi_{yz}+a_{33}\varphi_{xx}+a_{22}\varphi_{zz},\qquad(9)$$

where

$$D_1=\frac{v_H^2}{v_{6H}}\frac{v_{6H}^2-v_m^2}{\Omega^2},\qquad Q_1=\frac{7v_H^2}{v_{6H}},$$

$$2v_{6H}\alpha=-\left[v_H^2\left(\frac{6B_{14}}{b}+\frac{4v_{14}^2}{v_6^2}\right)-3v_{14}^2\right],$$

$$2v_{6H}a_{33}=-v_6^2-(v_1^2+v_{12}^2)\frac{6v_{14}^4-v_6^2(v_{6H}^2-v_4^2)}{2v_{14}^4+v_{12}^2(v_{6H}^2-v_4^2)}-v_{14}^2\frac{6v_{14}^2v_{12}^2+4v_{14}^2(3v_6^2+v_{14}^2)+2v_{12}^2(v_{6H}^2-v_4^2)}{2v_{14}^4+v_{12}^2(v_{6H}^2-v_4^2)},$$

$$2v_{6H}a_{22}=-\frac{4v_{14}^2}{v_6^2}-\frac{v_4^2}{2}+v_H^2\frac{v_4^2}{v_6^2}-8v_H^2\frac{B_{14}}{B}\frac{v_{14}^2}{v_6^2}.$$

Interestingly, as the field H is varied in hematite the coefficient a_{33} changes sign (a_{33} vanishes when H is near H_u, at which the condition $v_{6H}=v_4$ holds). It can be seen from (9) that with $\mathbf{v}\|y$ and $\mathbf{H}\|x$ the transverse perturbations are carried off along the z axis (the term $\alpha\varphi_{yz}$ can be eliminated by rewriting the equations in terms of $z - \alpha t$). We can derive yet another equation describing the evolution of waves which are propagating along the z axis. In this case the longitudinal sound is completely uncoupled with the magnetic oscillations, as is easily understood, and for transverse sound with linear magnetoelastic coupling we find the equation

$$\frac{\partial}{\partial z}(\varphi_t+D_2\varphi_{zzz}+Q_2\varphi^2\varphi_z)$$

$$=\frac{v_{14}^2}{2v_{4H}}\left(1+\frac{2v_4^2}{v_{4H}^2}\right)\varphi_{yz}+a_{33}\varphi_{xx}+a_{11}\varphi_{yy},$$

where

$$v_{4H}^2=v_4^2-\frac{8B_{14}^2H_E}{\rho M_0H(H+H_D)}=v_4^2-w_H^2,$$

$$D_2=\frac{w_H^2}{2v_{4H}}\frac{v_{4H}^2-v_m^2}{\Omega^2},\qquad Q_2=\frac{w_H^2}{8v_{4H}},$$

$$-2v_{4H}a_{33}=v_4^2\left[1-\frac{3v_{14}^4}{v_{4H}^2w_H^2}+\frac{v_6^2+4v_{14}^2}{v_{4H}^2}\right],$$

$$-2v_{4H}a_{11}=\frac{v_{14}^4}{v_{4H}^2}\left[4-2\frac{v_4^2}{v_{4H}^2}\left(2\frac{v_4^2}{v_{4H}^2}-1\right)\right.$$

$$\left.+\frac{v_4^2}{2v_{14}^4}(v_1^2+v_{13}^2)\right].$$

Both the coefficients on the left side of this equation and the very fact that there is no quadratic nonlinearity in this case are independent of the direction of the magnetic field in the basis plane. The coefficients on the right side are given for $\varphi_0 = \pi/4$. In this case, transverse perturbations are carried off along the y axis at a velocity $v_y \approx 3v_{14}^2/4v_{4H}$. The transverse drift which has been noted in all these cases imposes a lower limit on the transverse dimension of the crystal in attempts to observe self-focusing.

We conclude this section with the derivation of an equation for a more general case, containing a quadratic nonlinearity. For $\varphi_0 = \pi/8$ (in which case the coefficient of the quadratic term is at its maximum), we have $\mathbf{v}\|x$, $v^2 = v_1^2 - 1/2)v_H^2 = v_{1H}^2$, and the equation becomes, to within second derivatives with respect to the transverse coordinates,

$$\frac{\partial}{\partial x}\left(\varphi_t+\frac{v_H^2}{2v_{1H}}\frac{v_{1H}^2-v_m^2}{\Omega^2}\varphi_{xxx}-\frac{3v_H^2}{2v_{1H}}\varphi\varphi_x\right.$$

$$\left.+\frac{v_H^2}{2v_{1H}}\frac{H_D}{H+H_D}\varphi^2\varphi_x\right)=\frac{v_H^2}{2v_{1H}}(\xi_1\varphi_{xy}+\xi_2\varphi_{xz}),$$

$$\xi_1=1+\frac{3v_{14}^2}{b}\frac{v_{14}^2b+v_{12}^2B_{14}}{(v^2-v_4^2)v_{12}^2-4v_{14}^4}+\frac{v_1^2}{v_{12}^2}$$

$$+\frac{bv_{14}^2+v_{12}^2B_{14}}{(v^2-v_4^2)v_{12}^2-4v_{14}^2}\frac{4v_1^2v_m^2}{v_{12}^2b},$$

$$\xi_2=2\frac{B_{14}}{b}+\frac{2v_{14}^2(v_4^2+2v_{13}^2)}{v_{12}^2b}\frac{v_{14}^2b+v_{12}^2B_{14}}{(v^2-v_4^2)v_{12}^2-4v_{14}^4}$$

$$+\frac{v_4^2+2v_{13}^2}{2v_{12}^2}+\frac{3v_{14}^2}{b}\frac{v_{14}^2b+v_{12}^2B_{14}}{(v^2-v_4^2)v_{12}^2-4v_{14}^4}.$$

In this case, the transverse perturbations drift along both y and z, but the drift velocity in strong fields H is small in comparison with v_{14}, by a factor on the order of the parameter $(v_H/v_{1H})^2$.

We can transform from the equation for φ to one for u by means of the substitution

$$\varphi=-\frac{H_Eb}{M_0H(H+H_D)}\sqrt{2}\,u.$$

1.2. We now consider the propagation of sound waves in rare earth orthoferrites having the general formula $MFeO_3$, where M is a rare earth ion. The basic properties of these antiferromagnets are usually described by a two-sublattice model. The spin reorientation in orthoferrites occurs not at a single point on the temperature scale but over an interval of tens of degrees.[15] At temperatures T_1 and T_2 corresponding to the beginning and end of the reorientation, in the absence of an external magnetic field, two second-order phase transitions occur. For these crystals there is characteristically a soft mode (a low-frequency mode) of the antiferromagnetic resonance, so that the nonlinearity of the magnetoelastic waves in the ultrasonic range should be particularly pronounced, as was shown in Ref. 15.

The free-energy density of an orthoferrite is written as the sum of the magnetic, elastic, and magnetoelastic energy densities (see Ref. 15, for example):

$$F=F_m+F_e+F_{me},$$

$$F_m=2M_0[H_Em^2+H_D(m_xl_z-m_zl_x)+\frac{A_1}{2}l_x^2$$

$$+\frac{C_1}{2}l_z^2+\frac{A_2}{4}l_x^4+\frac{C_2}{4}l_z^4+\frac{G}{2}l_x^2l_z^2-(\mathbf{mH})],$$

$$F_e=\frac{1}{2}(C_{11}u_{xx}^2+C_{22}u_{yy}^2+C_{33}u_{zz}^2)+C_{12}u_{xx}u_{yy}$$

$$+C_{23}u_{yy}u_{zz}+C_{13}u_{zz}u_{xx}$$

$$+2C_{44}u_{yz}^2+2C_{55}u_{xz}^2+2C_{66}u_{xy}^2,$$

$$F_{me}=2[(B_{11}u_{xx}+B_{12}u_{yy}+B_{13}u_{zz})l_x^2+B_{55}u_{xz}l_xl_z$$
$$+(B_{21}u_{xx}+B_{22}u_{yy}+B_{23}u_{zz})l_y^2+B_{66}u_{xy}l_xl_y$$
$$+(B_{31}u_{xx}+B_{32}u_{yy}+B_{33}u_{zz})l_z^2+B_{44}u_{yz}l_yl_z].$$

Here A_1, C_1 and A_2, C_2, G are the bilinear- and biquadratic-anisotropy fields, and the notation is otherwise the same as in Subsection 1.1. The coordinates x, y, z run along the crystal axes a, b, c.

The most common reorientation transitions in orthoferrites, and those which have been studied most thoroughly, are transitions accompanied by a reorientation of the spins in the xz plane. In such crystals (e.g., $TmFeO_3$ and $SmFeO_3$), the vector $\mathbf{l}$ is directed along the crystal c axis at $T < T_1$, while at $T > T_2$ it is directed along the a axis. At intermediate temperatures, $T_1 < T < T_2$, the vector $\mathbf{l}$ rotates smoothly from one of these axes to the other. The equation for the oscillations of the vector $\mathbf{l}$ near the x axis (in the temperature region near T_2) is[10,15]

$$\gamma^{-2}[\mathbf{l},\Box_m\mathbf{l}]_y=2H_E[\mathbf{l},\mathbf{H}_l]_y,$$

$$\mathbf{H}_l=-\frac{1}{2M_0}\frac{\partial F}{\partial\mathbf{l}},\qquad\mathbf{l}=(\cos\varphi,0,\sin\varphi).\tag{10}$$

The acoustic oscillations are described by the equations of elastic theory, (5).

Let us examine waves which are propagating along the x axis. After some straightforward calculations, we find from (10) and (5)

$$\left(\frac{\partial^2}{\partial t^2}-v_m^2\frac{\partial^2}{\partial x^2}\right)\varphi=\lambda_1\sin 2\varphi+\lambda_2\sin 4\varphi+\mu_1 u\sin 2\varphi-\mu_2 w\cos 2\varphi, \quad (11a)$$

$$\left(\frac{\partial^2}{\partial t^2}-v_1^2\frac{\partial^2}{\partial x^2}\right)u=\frac{\partial^2}{\partial x^2}(\cos 2\varphi-1), \quad (11b)$$

$$\left(\frac{\partial^2}{\partial t^2}-v_5^2\frac{\partial^2}{\partial x^2}\right)w=\frac{\partial^2}{\partial x^2}\sin 2\varphi, \quad (11c)$$

where

$$u=\frac{\rho}{B_{11}-B_{13}}u_{xx},\qquad w=\frac{2\rho}{B_{55}}u_{xz},$$

$$v_5^2=\frac{2C_{55}}{\rho}\qquad v_1^2=\frac{C_{11}}{\rho},$$

$$\lambda_1=\gamma^2H_E\left(A_1-C_1+\frac{A_2}{2}-\frac{C_2}{2}\right),$$

$$\lambda_2=\frac{\gamma^2H_E}{4}(C_2+A_2-2G),\qquad \mu_1=\frac{2\gamma^2H_E}{\rho M_0}(B_{11}-B_{31})^2,$$

$$\mu_2=\frac{\gamma^2H_E}{\rho M_0}B_{55}^2.$$

We first consider small-amplitude steady-state solutions of (11). Expressing u and w in terms of φ, we find one equation for φ:

$$\varphi_{\xi\xi}=A\varphi-B\varphi^3,$$

where $\xi=x-vt$.

The soliton solution is of the form

$$\varphi=\varphi_0\,\mathrm{sech}\left(\frac{\xi-\xi_0}{\Delta}\right),\qquad \varphi_0=\left(\frac{2A}{B}\right)^{1/2},\qquad \Delta=A^{-1/2}.$$

The condition for the applicability of the expansion in φ is $2A\ll B$. As in Subsection 1.1, we have two different possibilities here: solitons propagating at velocities $v\approx v_1$, in which case the reduced equations describing the motions are the same as (2); and solitons moving at velocities

$$v^2\approx v_{5H}^2=v_5^2+\frac{B_{55}^2}{\rho M_0(A_1+A_2-C_1-G)},$$

in which case the truncated equation is the modified Korteweg–de Vries equation (1), (7), with the coefficients

$$a_{11}=v_{5H}v_4^2/v_5^2,\qquad a_{12}=0,$$

$$a_{22}=\frac{v_{5H}}{2}\left(1+\frac{v_{13}^2+v_{33}^2}{v_5^2}-\frac{v_1^2+v_{13}^2}{v_1^2-v_5^2}\right),$$

$$D=\frac{(v_m^2-v_{5H}^2)B_{55}^2}{4v_{5H}\rho M_0\gamma^2H_E(A_1+A_2-C_1-G)^2},$$

$$Q=\frac{3B_{55}^2}{v_{5H}\rho M_0(A_1+A_2-C_1-G)}\left[A_1-C_1+\frac{A_2}{2}-\frac{C_2}{2}+\frac{2(B_{11}-B_{31})^2}{\rho M_0(v_1^2-v_{5H}^2)}\right].$$

As the temperature is varied, Q may change sign, and if so there will be a substantial change in the nonlinear dynamics of the waves (§2).

One-dimensional equations describing wave propagation along the z axis are found from (11) through the substitutions $\partial/\partial x\to\partial/\partial z, v_1^2\to v_3^2=C_{33}/\rho$. We then have

$$u=\frac{\rho}{B_{13}-B_{33}}u_{zz},\qquad \mu_1=\frac{2\gamma^2H}{\rho M_0}(B_{13}-B_{33})^2.$$

Consequently, all the conclusions reached in the preceding subsection remain valid for this wave propagation direction.

If the velocity is along the y axis, there are no sound waves which are linearly coupled with magnons, so that weakly nonlinear waves propagating in a single direction are described by system (2).

We also note that there exist exact steady-state solutions of Eqs. (11) for which the condition $\varphi\ll 1$ generally is not necessary (see also Ref. 16):

$$\varphi=\mathrm{arctg}\left[\frac{A}{\mathrm{ch}\,s}\right],\qquad u=\frac{2}{(v_1^2-v^2)}\frac{A^2}{[\mathrm{ch}^2 s+A^2]},$$

$$w=\frac{2A\,\mathrm{ch}\,s}{(v^2-v_5^2)[\mathrm{ch}^2 s+A^2]},$$

where $\xi=x-vt$, $s=(\xi-\xi_0)/\Delta$.

$$\Delta^{-2}=\frac{2\gamma^2H_E}{v_m^2-v^2}\left[C_1+G-A_1-A_2+\frac{B_{55}^2}{\rho M_0(v^2-v_5^2)}\right],$$

$$A^2=\left[C_1+G-A_1-A_2+\frac{B_{55}^2}{\rho M_0(v^2-v_5^2)}\right]\times\left[A_1-C_1+\frac{A_2}{2}-\frac{C_2}{2}+\frac{2(B_{11}-B_{13})}{\rho M_0(v_1^2-v^2)}\right]^{-1}. \quad (12)$$

The intervals of possible velocities v of solutions (12) are determined from the conditions $\Delta^2>0$ and $A^2>0$. In the limit $A\to 0$ we find small-amplitude solitons from (12).

These examples show that, despite the wide variety of physical situations associated with the anisotropy of a crystal, the dynamics of slightly nonlinear waves displays a universality and can be described by either Eq. (1) or the system of equations (2). It is thus worthwhile to study the properties of these universal equations in more detail, and this is the purpose of the following section of this paper.

§2. STABILITY OF THE SOLITONS AND SELF-FOCUSING THEOREM

As was shown in the preceding section, the evolution of phonons which are not linearly coupled with magnons is described by the following system of equations in a coordinate system moving at the sound velocity v_s:

$$\frac{v_m^2-v_s^2}{\gamma^2}\varphi_{xx}=H(H+H_D)\varphi+\frac{2H_Eb}{M_0}\varphi u,\qquad u_t+g\varphi\varphi_x=0, \quad (13)$$

where $g\approx b/\rho v_s$ [see, for example, (8)]. It is easy to see that this system of equations has soliton solutions $u_0(x+vt),\varphi_0(x+vt)$ only if $v_s<v_m$:

$$\varphi_0=\left(\frac{v2M_0H(H+H_D)}{H_Ebg}\right)^{1/2}\frac{1}{\mathrm{ch}\,\varkappa(x+vt)},$$

$$u_0=-\frac{M_0H(H+H_D)}{H_Eb\,\mathrm{ch}^2\varkappa(x+vt)},$$

where $\varkappa^2=\Omega^2/(v_m{}^2-v_s{}^2)$. The magnitude of the strain tensor corresponding to the soliton solution does not depend on the velocity v; for hematite with $\varphi_0=\pi/4$ and for $H=1$ kOe it is

$$u_{xz}=\frac{b}{2B_{14}}\frac{M_0H(H+H_D)}{H_Eb}=4\cdot10^{-5}.$$

In the case $v_s>v_m$, the absence of solitons—i.e., the absence of localized steady-state solutions for which the nonlinearity and the dispersion cancel each other out—apparently means that small initial perturbations spread out. In the case $v_s<v_m$ we find a different and more interesting evolution. Before we analyze this case, it is convenient to switch to the dimensionless variables

$$x\to\varkappa x,\quad u\to\frac{M_0H(H+H_D)}{2H_Eb}u,$$

$$\varphi^2\to v_s\frac{M_0H(H+H_D)}{H_Ebg}\varphi^2,\quad t\to\left(\frac{v_m{}^2-v_s{}^2}{v_s{}^2\Omega^2}\right)^{1/2}t.$$

We then find the system of equations

$$u_t+\varphi\varphi_x=0,\quad \varphi_{xx}=\varphi+u\varphi.$$

In these dimensionless variables, the soliton solutions become

$$u_0=-\frac{2}{\operatorname{ch}^2(x+vt)},\quad \varphi_0=\frac{2v^{1/2}}{\operatorname{ch}(x+vt)}. \qquad (14)$$

We see that the solitons are moving at a velocity below the velocity of sound.

The stability of steady-state solutions (14) is crucial to an understanding of the dynamics of sound waves within the framework of system (13).

We linearize (13) about a soliton,

$$pf+vf_\xi+\frac{\partial}{\partial\xi}(\varphi_0g)=0,\quad g_{\xi\xi}-g-u_0g=\varphi_0f, \qquad (15)$$

for a perturbation

$$\delta u=f(\xi)e^{pt},\quad \delta\varphi=g(\xi)e^{pt},\quad \xi=x+vt.$$

The spectral problem (15) can be solved exactly by the isospectral-transformation of Ref. 13. For simplicity, we omit the calculations, proceeding immediately to the final result. The continuum functions are

$$f_k=e^{ik\xi}\sum_{n=1}^{3}a_n\operatorname{th}^n\xi,$$

$$g_k=e^{ik\xi}\frac{v^{1/2}}{\operatorname{ch}\xi}\left(-\frac{ik}{2}+\operatorname{th}\xi\right), \qquad (16)$$

$$p=ikv,\quad a_0=ik\left(1+\frac{k^2}{4}\right),$$

$$a_1=-(2+k^2),\quad a_2=-2ik,\quad a_3=2,$$

as can be verified by direct substitution. For $k=0$ we find from (16) a shear mode of neutral stability:

$$f_0=-\frac{2\operatorname{th}\xi}{\operatorname{ch}^2\xi},\quad g_0=\frac{v^{1/2}\operatorname{th}\xi}{\operatorname{ch}\xi},\quad p=0.$$

With $k=-2i$ we find from (16) a discrete-spectrum function, which describes the linear stage of the instability of the soliton solution (14):

$$f_1=\frac{2\operatorname{th}\xi}{\operatorname{ch}^2\xi},\quad g_1=\frac{v^{1/2}}{\operatorname{ch}\xi}(1-\operatorname{th}\xi),\quad p_1=2v.$$

The reason for this instability can be explained qualitatively as follows. System (13) has no linear dispersion (since there is no linear magnetoelastic coupling). Although it is possible to construct a steady-state solution (14) in which the nonlinearity and the (also nonlinear) dispersion cancel out, there exist perturbations of solitons for which the nonlinear effects are predominant. The probable outcome of the evolution of such an instability within the framework of (13) is the formation of a singularity over a finite time, i.e., a collapse. In this case we are of course going beyond the range of applicability of system (13), so that this result cannot have a literal physical meaning. The important point here is the very possibility that the amplitudes can grow to this extent, with the nonlinearity no longer remaining small. The subsequent evolution of an unstable perturbation must be described by means of the complete equations, of the type (11). It might be hypothesized that the value u_0 used for the soliton solution is a threshold value: Initial perturbations with $u<u_0$ spread out, while those with $u>u_0$ break.

As was shown in §1, the equation describing the evolution of a system of phonons which are coupled linearly with magnons can be written as follows in terms of dimensionless variables:

$$\frac{\partial}{\partial x}(u_t+12u^2u_x+Du_{xxx})=3b\Delta_\perp u. \qquad (17)$$

This equation is written in a coordinate system which is moving at the sound velocity v_s. In it we have retained the basic terms (after the elimination of the velocity v_s) which are responsible for the weak nonlinearity ($\sim u^2u_x$), the weak dispersion ($\sim u_{xxx}$), and the diffractive divergence $\sim\Delta_\perp u$). Here D and b take on the values ±1, depending on the signs of the dispersion along the longitudinal and transverse coordinates, respectively. Equation (17) is a general equation describing sound waves with a weak dispersion when the coefficient of the quadratic term vanishes by virtue of the symmetry of the system.

Equation (17) is one of Hamilton's equations

$$u_t=\frac{\partial}{\partial x}\frac{\delta\mathscr{H}}{\delta u},$$

for the Hamiltonian

$$\mathscr{H}=\int\left[\frac{D}{2}u_x{}^2+\frac{3b}{2}(\nabla_\perp w)^2-u^4\right]d\mathbf{r},$$

where $w_x=u$. Equation (17) conserves, in addition to $\mathscr{H}$, the momentum $P=\int u^2d\mathbf{r}$.

With $D=-1$ in (17), the nonlinearity cannot cancel the dispersion, since it (like the dispersion) causes a spreading of a wave packet. Consequently, there are no one-dimensional soliton solutions for this sign of the longitudinal dispersion.

In the case $b=1$, the one-dimensional soliton solutions of (17) are unstable with respect to bending vibrations of the front.[17] Using the method of Ref. 18, we can show that Eq. (17) has no stable three-dimensional solitons. Consequently, an arbitrary initial distribution may either spread out or collapse. The nature of the evolution is determined by the sign of the Hamiltonian, $\mathscr{H}$, as we will now show.

A distribution with a negative Hamiltonian cannot spread out in a dispersive manner, since a simple analysis shows that at small values of u the Hamiltonian $\mathscr{H}$ is positive. It is natural to conclude that the evolution of an initial condition of this sort leads to a singularity. Within the framework of Eqs. (17) it is possible to determine an exact sufficient condition for beam self-focusing.

Theorem. A solution of Eq. (17) for which the condition $\mathscr{H} < 0$ holds at the initial time becomes singular in a finite time.

Proof. We consider the positive quantity $\int r_\perp{}^2 u^2 d\mathbf{r}$, which is proportional to the square of the characteristic dimension of the beam. We calculate the second derivative of this quantity with respect to the time. After some straightforward calculations, we find

$$\frac{\partial^2}{\partial t^2}\int r_\perp{}^2 u^2\, d\mathbf{r} = 48\mathscr{H} - 24Db\int u_x{}^2\, d\mathbf{r}. \tag{18}$$

With $D = b = +1$ we thus have

$$\int r_\perp{}^2 u^2\, d\mathbf{r} \leqslant 24\mathscr{H}t^2 + R_1 t + R_2,$$
$$R_1 = \frac{\partial}{\partial t}\int r_\perp{}^2 u^2\, d\mathbf{r}\,|_{t=0}, \quad R_2 = \int r_\perp{}^2 u^2\, d\mathbf{r}. \tag{19}$$

If $\mathscr{H} < 0$, inequality (18) must break down at some finite t:

$$t < \tau = -\left[\frac{R_1}{2} - \left(\frac{R_1{}^2}{4} + 48\mathscr{H}R_2\right)^{1/2}\right](24\mathscr{H})^{-1}. \tag{20}$$

This result means that a singularity appears in the solution in a finite time. Relation (20) leads to an upper limit on the distance over which beam self-focusing occurs.

If the sign of at lease one of the quantities D, b is negative, Eq. (18) shows that the beam undergoes defocusing. Specifically, if $D = -1$ and $b = +1$, a sufficient condition for the spreading of the beam is $\mathscr{H} > 0$, while if $D = +1$ and $b = -1$ this condition is $\mathscr{H} < 0$. With $D = b = -1$ we have

$$\frac{\partial^2}{\partial t^2}\int r_\perp{}^2 u^2\, d\mathbf{r} = 48\left[\frac{3}{2}\int (\nabla_\perp w)^2\, d\mathbf{r} + \int u^4\, d\mathbf{r}\right] > 0.$$

We should point out that this theorem holds for an arbitrary nonlinearity $u^{n-2}u_x$ with $n \geqslant 2(d+1)/(d-1)$, where d is the dimensionality of the space.

Like the nonlinear Schrödinger equation, this condition is sufficient but not necessary. Using the substitution

$$u = \psi e^{i\lambda x} + \psi^* e^{-i\lambda x},$$

and taking an average over the fast oscillations, we find the famous Vlasov-Petrishchev-Talanov theorem[19] as a particular case of the theorem which we have just proved, since this substitution causes the envelope $\psi(r_\perp, t)$ to obey the nonlinear Schrödinger equation.

CONCLUSION

In summary, we can expect to be able to observe stable solitons only in substances with $v_s > v_m$ ($MnCO_3$, $RbMnF_3$) and for sound waves with a linear magnetoelastic coupling [it is also necessary to ensure that a_{11}, a_{22}, and a_{33}, given by (7) or (9), are negative]. Self-focusing due to a quadratic nonlinearity can be observed in hematite (α-Fe_2O_3) with $\varphi_0 \neq \pi n/4$, while self-focusing due to a cubic nonlinearity can be observed in $MFeO_3$ (where M is a rare earth ion) or $MnCO_3$ (if $a_{ii} > 0$ and $\varphi_0 = \pi n/4$). The one-dimensional breaking effects described by system (13) can be observed in hematite at strain values which are not linearly coupled with magnetic oscillations and which are comparable to $M_0 H(H + H_D)/H_E b$.

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[1]V. I. Ozhogin, Izv. Akad. Nauk SSSR, Ser. Fiz. **42** 1625 (1978) [Bull. Acad. Sci. USSR Phys. Series **42**, (8), 48 (1978)].
[2]V. I. Ozhogin and V. P. Preobrazhenskiĭ, Zh. Eksp. Teor. Fiz. **73**, 988 (1977) [Sov. Phys. JETP **46**, 523 (1977)].
[3]V. I. Ozhogin, A. Yu. Lebedev, and A. Yu. Yakubovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 333 (1978) [JETP Lett. **27**, 313 (1978)].
[4]A. Yu. Lebedev, V. I. Ozhogin, and A. Yu. Yakubovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 22 (1981) [JETP Lett. **34**, 19 (1981)].
[5]L. K. Zarembo and V. A. Krasil'nikov, Usp. Fiz. Nauk **102**, 549 (1970) [Sov. Phys. Usp. **13**, 778 (1971)].
[6]V. L. Preobrazhenskiĭ, M. A. Savchenko, and N. A. Ékonomov, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 93 (1978) [JETP Lett. **28**, 87 (1978)].
[7]V. V. Berezhnov, N. N. Evtikhiev, V. L. Preobrazhenskiĭ, and N. A. Ékonomov, Akust. Zh. **26**, 328 (1980) [Sov. Phys. Acoust. **26**, 180 (1980)].
[8]E. A. Kuznetsov, S. L. Musher, and A. V. Shafarenko, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 204 (1983) [JETP Lett. **37**, 241 (1983)].
[9]V. I. Ozhogin and A. Yu. Lebedev, J. Magn. Magn. Mater. **15-18**, 617 (1980).
[10]V. I. Ozhogin, D. Yu. Main, V. I. Petviashvili, and A. Yu. Lebedev, IEEE Trans. Magn. **Mag-19**, 1977 (1983).
[11]V. I. Karpman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, Moscow, 1973.
[12]B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR **192**, 753 (1970) [Sov. Phys. Dokl. **15**, 539 (1970)].
[13]E. A. Kuznetsov, M. D. Spector, and G. E. Fal'kovich, Physica **D10**, 379 (1984).
[14]E. A. Kuznetsov and S. K. Turitsyn, Zh. Eksp. Teor. Fiz. **82**, 1457 (1982) [Sov. Phys. JETP **55**, 844 (1982)].
[15]A. Yu. Lebedev, V. I. Ozhogin, V. L. Safonov, and A. Yu. Yakubovskiĭ, Zh. Eksp. Teor. Fiz. **85**, 1059 (1983) [Sov. Phys. JETP **58**, 616 (1983)].
[16]V. P. Preobrazhensky and M. A. Savchenko, in: *Digests of Twentieth Congress Ampere, Tallin*, 1979, p. 410.
[17]A. Newell, in: *Solitons* (Russ. Transl. Mir, Moscow, 1983).
[18]V. E. Zakharov, E. A. Kuznetsov, and A. M. Rubenchik, Preprint No. 199, Institute of Automation and Electrometry, Novosibirsk, 1983.
[19]V. N. Vlasov, I. A. Petrishchev, and V. I. Talanov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **14**, 1352 (1971).

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