Stability of vector solitons in optical fibers

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The stability of vector solitons is considered in the framework of two coupled nonlinear Schrödinger equations. It is found that a bound state in the form of two coupled bright solitons achieves a minimum Hamiltonian for a fixed number of quasi-particles, and thus the stability of such a solution may be proved by direct construction of the Lyapunov function. An integral condition is obtained for an initial wave packet under which the fractional pulses are stabilized against splitting.

The copropagation of two high-intensity optical pulses in a nonlinear medium with Kerr-type nonlinearity is governed by the incoherently coupled nonlinear Schrödinger equations:

$$i(U_z + \delta U_t) + \frac{\beta_1}{2} U_{tt} + \gamma_1(|U|^2 + \alpha|V|^2)U = 0, \quad (1)$$

$$i(V_z - \delta V_t) + \frac{\beta_2}{2} V_{tt} + \gamma_2(|V|^2 + \alpha|U|^2)V = 0, \quad (2)$$

where U and V are the slowly varying amplitudes of the wave envelopes, $\beta_{1,2}$ are the group-velocity dispersions, α and $\gamma_{1,2}$ are the nonlinear coefficients describing the wave self-phase modulation and the cross-phase modulation, respectively, and 2δ corresponds to the group-velocity detuning. Equations (1) and (2) describe both the evolution of two coupled waves with the same polarizations but different frequencies and the propagation of two waves with the same frequency but different polarizations. In this Letter attention is paid mainly to optical pulse propagation in the anomalous dispersion regime of a birefringent fiber, but our results can also be applied to a wide array of different physical problems modeled by Eqs. (1) and (2).

The nonlinear interaction of two polarizations, which results from the tensor character of the susceptibility of nonlinear-optical media, has recently attracted substantial attention.[1–12] As was noted in Refs. 1 and 2, isotropic single-mode fibers are really bimodal because of the presence of a small intrinsic birefringence that can lead to the splitting of the pulses with the different polarizations. The small birefringence is caused partially by the ellipticity of the core of the optical fiber that breaks the radial symmetry and partially by material contribution. This effect can substantially limit the possibilities of using such fibers for the purpose of information transmission. However, as was shown by Menyuk,[2,3] the Kerr effect can stabilize solitons against splitting that is due to birefringence. Either of the two polarization modes is able to capture the other one such that the two pulses can propagate together in spite of the group-velocity mismatch. The details of the application of Eqs. (1) and (2) to real fibers were considered in Refs. 1 and 3, so in this Letter we concentrate only on the mathematical aspects of the problem. By using the well-known variables expressed in soliton units,[4] Eqs. (1) and (2) may be written in dimensionless form:

$$i(U_z + \delta U_t) + \frac{1}{2} U_{tt} + (|U|^2 + \alpha|V|^2)U = 0, \quad (3)$$

$$i(V_z + \delta V_t) + \frac{1}{2} V_{tt} + (|V|^2 + \alpha|U|^2)V = 0. \quad (4)$$

In the case of silica or similar fibers, where the dominant contribution to the $\lambda^{(3)}$ is of electronic origin,[12] the coefficient α is $2/3 < \alpha < 2$ for elliptical eigenmodes, $\alpha = 2/3$ for linearly polarized modes, and $\alpha = 2$ for circular polarized modes.

Equations (3) and (4) have been studied numerically[2,3] and by using a variational approach.[9,10] It was shown that for small amplitudes the two pulses separate owing to the different group velocities. Above a certain amplitude threshold a fraction of the energy in one polarization is captured by the other mode, and solitons consisting of both polarizations are formed. The amplitude threshold increases with the strength of the birefringence parameter δ. The bound state with the mixed polarization results when the partial pulses in each polarization shift their phases in such a way that their group velocities become equal.

Contrary to the nonlinear Schrödinger equation, Eqs. (3) and (4) cannot be solved by using the inverse scattering transform. Therefore many important questions concerning the soliton dynamics in the framework of Eqs. (3) and (4) are still open. For example, which kinds of initial field distribution lead to the appearance of a vector soliton?

In this Letter we prove the stability of the vector solitons for the case of equal amplitudes in each

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polarization and obtain sufficient integral conditions for the mutual trapping of fractional pulses.

Equations (3) and (4) have a Hamiltonian form:

$$iU_z = \frac{\delta H}{\delta U^*}, \tag{5}$$

$$iV_z = \frac{\delta H}{\delta V^*}, \tag{6}$$

with the Hamiltonian

$$H = \frac{i}{2}\delta \int (UU_t^* - U^*U_t + V^*V_t - VV_t^*)\mathrm{d}t + \frac{1}{2}\int (|U_t|^2 + |V_t|^2)\mathrm{d}t - \frac{1}{2}\int (|U|^4 + |V|^4 + 2\alpha|U|^2|V|^2)\mathrm{d}t.$$

In addition to H, Eqs. (3) and (4) have the following integrals of motion: the momentum $P = i/2 \int (UU_t^* - U^*U_t + VV_t^* - V^*V_t)\mathrm{d}t$ and the number of quasi-particles in each polarization $N_1 = \int |U|^2\mathrm{d}t$ and $N_2 = \int |V|^2\mathrm{d}t$. The only conserved quantities relevant to the stability of solutions of master equations (3) and (4) are the Hamiltonian and N_1 and N_2. The momentum can be set to an arbitrary value by using the symmetries of equations.

We investigate the stability of a stationary solution of Eqs. (3) and (4) of the form

$$U_0(z,t) = \frac{1}{\sqrt{(1+\alpha)}}\exp\left[\frac{i}{2}(1+\delta^2)z - i\delta t\right]\mathrm{sech}(t), \tag{7}$$

$$V_0(z,t) = \frac{1}{\sqrt{(1+\alpha)}}\exp\left[\frac{i}{2}(1+\delta^2)z + i\delta t\right]\mathrm{sech}(t). \tag{8}$$

The Hamiltonian H takes the following value for soliton solutions:

$$H[U_0, V_0] = -\frac{\delta^2}{2}(N_1 + N_2) - \frac{(1+\alpha)^2}{24}(N_1^3 + N_2^3).$$

The soliton solutions (7) and (8) of Eqs. (3) and (4) can be viewed as solutions of the Euler equation that corresponds to the following variational problem:

$$\delta S = \delta(H + \lambda^2 N) = 0. \tag{9}$$

Equation (9) means that all localized stationary solutions of Eqs. (3) and (4) realize the stationary points of the Hamiltonian H for the fixed N.

To prove the soliton stability it is sufficient to present the Lyapunov function L, which satisfies the following conditions (see, e.g., Refs. 13–16):

(1) $L[U_0, V_0] = 0$, a minimum of L is attained on the stationary solutions.
(2) $L[U, V] \geq 0$, L is a nonnegative functional for the perturbed states.
(3) $\mathrm{d}L/\mathrm{d}z \leq 0$, L is a nonincreasing function of z.

The present method of stability investigation is naturally formulated for Hamiltonian systems.[13] If the Hamiltonian of the system H is bounded for the fixed N, one may use as the Lyapunov function the combination $L = H - \min(H)$.

We now demonstrate the boundedness of H for a fixed N. It is suitable to present the functions U and V in the form $U = f\exp(i\theta_1)$ and $V = g\exp(i\theta_2)$. Inserting these expressions for U and V into the Hamiltonian, we may rewrite H as

$$H = \delta\int (f^2\theta_{1t} - g^2\theta_{2t})\mathrm{d}t + \frac{1}{2}\int (f^2{\theta_{1t}}^2 + g^2{\theta_{2t}}^2)\mathrm{d}t + \frac{1}{2}\int ({f_t}^2 + {g_t}^2)\mathrm{d}t - \frac{1}{2}\int (f^4 + g^4 + 2\alpha f^2g^2)\mathrm{d}t. \tag{10}$$

One may easily estimate the first two terms in the Hamiltonian,

$$\delta\int (f^2\theta_{1t} - g^2\theta_{2t})\mathrm{d}t + \frac{1}{2}\int (f^2{\theta_{1t}}^2 + g^2{\theta_{2t}}^2)\mathrm{d}t = \frac{1}{2}\int [f^2(\delta + \theta_{1t})^2 + g^2(\delta - \theta_{2t})^2]\mathrm{d}t - \frac{\delta^2}{2}\int (f^2 + g^2)\mathrm{d}t \geq -\frac{\delta^2}{2}(N_1 + N_2). \tag{11}$$

At the next step, we use the following inequality (see, e.g., Ref. 17 for details) to estimate the nonlinear term:

$$\int f^4\mathrm{d}t \leq \frac{1}{\sqrt{3}}\left(\int f_t^2\mathrm{d}t\right)^{1/2}\left(\int f^2\mathrm{d}t\right)^{3/2}. \tag{12}$$

Substitution of these inequalities into the Hamiltonian (7), after simple algebra, leads to the formula

$$H \geq -\frac{\delta^2}{2}(N_1 + N_2) - \frac{(1+\alpha)^2}{24}(N_1^3 + N_2^3). \tag{13}$$

The remarkable fact is that the minimum of the Hamiltonian H, which we have found, is reached exactly on the soliton solutions (7) and (8): $H[U_0, V_0] = \min H$. This means that the soliton solutions (7) and (8) achieve the minimum of the H for the fixed N, and so the Lyapunov function L for the bound state in the form of solutions (7) and (8) can be constructed.

Because Eqs. (3) and (4) are not integrable (in general), there is no exact answer even to the following simply formulated question: Which kind of initial conditions for Eqs. (3) and (4) leads to the establishment of vector soliton (7) and (8)? In order to obtain the working criteria of the creation of vector solitons from the arbitrary initial pulse in the framework of Eqs. (3) and (4), let us consider the following equality:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\int t^2(|U|^2 + |V|^2)\mathrm{d}t = 2H + 2\delta^2 N + \int (|U_t|^2 + |V_t|^2)\mathrm{d}t + i\delta\int (UU_t^* - U^*U_t - VV_t^* + V^*V_t)\mathrm{d}t. \tag{14}$$

The last term in Eq. (14) can be estimated as above. Inserting estimation (11) into Eq. (14), we get

$$\frac{d^2}{dz^2}\int t^2(|U|^2 + |V|^2)dt \geq 2H + \delta^2 N,$$

the sufficient condition of spreading or splitting of the pulses for simple configurations. The term simple means that, in each polarization, distribution with only one maximum exists. For such configurations, under the condition $2H + \delta^2 N > 0$, the integral $\int t^2(|U|^2 + |V|^2)dt$ tends to infinity as $(H + 1/2\delta^2 N)z^2$, which corresponds either to dispersional spreading of initial distributions or to splitting apart of the partial pulses in each of the two polarizations. In both situations a vector soliton does not appear. For the initial conditions used in Refs. 2 and 3 $[U(0,t) = V(0,t) = A\ \mathrm{sech}(t)]$ the inequality $2H + \delta^2 N \geq 0$ gives the following estimate of the amplitude threshold: for $A \leq A_{cr} = \sqrt{0.6 + 1.8\delta^2}$ a vector soliton does not form. Evidently, for the more complicated configurations one need to use another approach.

Therefore, we now obtain an exact condition for the mutual trapping of the pulses from different polarizations. It would be reasonable to define the following criterion of a cross capture in the framework of Eqs. (3) and (4). One may say about a capture then that the cross integral $\int |U|^2|V|^2 dt$ is bounded from below by some positive constant.

In order to find the criteria of a cross capture, we rewrite the expression for the Hamiltonian:

$$\begin{aligned} 2\alpha \int |U|^2|V|^2 dt &= -H + \frac{i}{2}\delta \int \left(UU_t^* - U^*U_t + V^*V_t - VV_t^*\right)dt \\ &\quad + \frac{1}{2}\int (|U_t|^2 + |V_t|^2)dt - \frac{1}{2}\int (|U|^4 + |V|^4)dt \\ &\geq -H - \frac{\delta^2}{2}N + \frac{1}{2}\int (f_t^{\,2} + g_t^{\,2})dt \\ &\quad - \frac{1}{2}\int (f^4 + g^4)dt. \end{aligned} \tag{15}$$

Noting that the nonlinear term in the right-hand side of relation (15) can be estimated by using formula (11), we easily get $(I_1 = \int f_t^{\,2}dt,\ I_2 = \int g_t^{\,2}dt)$

$$\begin{aligned} 2\alpha \int |U|^2|V|^2 dt &\geq -H - \frac{\delta^2}{2}N + \frac{1}{2}(I_1 + I_2) \\ &\quad - \frac{1}{2\sqrt{3}}I_1^{1/2}N_1^{3/2} - \frac{1}{2\sqrt{3}}I_2^{1/2}N_2^{3/2} \\ &\geq -H - \frac{\delta^2}{2}N - \frac{1}{24}(N_1^3 + N_2^3). \end{aligned} \tag{16}$$

Thus, for the arbitrary initial distribution with

$$H < -\frac{\delta^2}{2}N - \frac{1}{24}(N_1^3 + N_2^3),$$

the cross integral $\int |U|^2|V|^2 dt$ is bounded from below by some positive constant. This means that some part of energy in one polarization is captured by another during the propagation along the fiber. We recognize that this integral criterion is fairly rough, but nevertheless it can be used, at least for a sufficiently small δ.

We would like to point out that the criterion obtained in our Letter can be used for an initial pulse with arbitrary shape, not only for the sech-type pulses, which were considered in Refs. 2, 3, 9, and 10.

In conclusion, we have proved the stability of vector solitons by constructing the Lyapunov function. We have found analytically the integral requirements on the parameters of the injected pulses that are sufficient for mutual trapping of fractional pulses of different polarizations.

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