Instability of two-dimensional solitons in discrete systems

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An exact analytical criterion for the instability of discrete soliton solutions of the two-dimensional discrete nonlinear Schrödinger equation is presented. Two-dimensional discrete oscillating breather-like solutions are also examined. © *1995 American Institute of Physics.*

Nonlinear discrete models have been studied in a variety of physical problems such as optical pulse propagation in arrays of coupled optical waveguides,[1–4] proton dynamics in hydrogen-bonded chains, the Davydov and Holstein mechanisms for the transport of excitation energy in biophysical systems, Scheibe aggregates,[5] the Hubbard model, and electrical lattices,[6,7] DNA dynamics,[8–10] and molecular vibrations in benzene chains, for example, and in certain molecular crystals.[11] Many common features of different discrete nonlinear models can be understood on the basis of the discrete nonlinear Schrödinger equation (DNLSE).[12–15] Two-dimensional localized discrete solutions of the DNLSE have been investigated in Ref. 15. The central question for the physical relevance of discrete solitons is their stability. Soliton stability plays a key role in the problem of the formation of large-amplitude states in *multi-dimensional discrete nonlinear systems.* The stability problem in the continuum limit of the DNLSE (two-dimensional NLSE) was solved in Ref. 16. In this paper we present an exact analytical criterion for discrete soliton instability in the two-dimensional DNLSE.

The two-dimensional discrete DNLSE is written

$$i\,\frac{\partial\Psi_{n,m}}{\partial t}+\Psi_{n+1,m}+\Psi_{n-1,m}+\Psi_{n,m+1}+\Psi_{n,m-1}-4\Psi_{n,m}+2|\Psi_{n,m}|^2\Psi_{n,m}=0. \qquad (1)$$

We have considered two types of boundary conditions on Eq. (1): (i) periodic boundary conditions $\Psi_{-N,m}=\Psi_{N,m}, \Psi_{n,-M}=\Psi_{n,M}$ and (ii) $|\Psi_{n,m}|\rightarrow 0$ for $\sqrt{n^2+m^2}\rightarrow\infty$. Here $2N$ and $2M$ are the numbers of the modes in the n and m directions, respectively.

Equation (1) can be written in the Hamiltonian form

$$i\,\frac{\partial\Psi_{n,m}}{\partial t}=\frac{\delta H}{\delta\Psi^{*}_{n,m}}, \qquad (2)$$

with Hamiltonian

$$H=\sum_{n,m}|\Psi_{n,m}-\Psi_{n-1,m}|^2+\sum_{n,m}|\Psi_{n,m}-\Psi_{n,m-1}|^2-\sum_{n,m}|\Psi_{n,m}|^4=I_1+I_2-I_3. \quad (3)$$

The integral $P=\Sigma|\Psi_{n,m}|^2$ is an additional conserved quantity.

Let us consider steady-state solutions, having the time dependence $\Psi_{n,m}=F_{n,m}\exp(i\lambda^2 t)$. The shape of $F_{n,m}$ is then determined by

$$F_{n+1,m}+F_{n-1,m}+F_{n,m+1}+F_{n,m-1}-(4+\lambda^2)F_{n,m}+2|F_{n,m}|^2F_{n,m}=0. \quad (4)$$

Soliton solutions correspond to extrema of H for fixed P, since Eq. (4) can be written as $\delta(H+\lambda^2P)=0$. Soliton solutions of Eq. (4) have been studied in Ref. 15.

In this letter we consider only the instability side of the stability problem. Details of a comprehensive investigation dealing with the stable solitons will be published elsewhere.

Linearizing Eq. (1) with respect to the soliton solution $[\Psi_{n,m}=(F_{n,m}+f_{n,m}+ig_{n,m})\exp(i\lambda^2t)]$ and decomposing the perturbations into real and imaginary parts, we obtain equations for the evolution of the real functions $f_{n,m}$ and $g_{n,m}$:

$$-\frac{dg_{n,m}}{dt}=-f_{n+1,m}-f_{n-1,m}-f_{n,m+1}-f_{n,m-1}+(4+\lambda^2)f_{n,m}-6F^2_{n,m}f_{n,m}=H_-f_{n,m} \quad (5)$$

and

$$\frac{df_{n,m}}{dt}=-g_{n+1,m}-g_{n-1,m}-g_{n,m+1}-g_{n,m-1}+(4+\lambda^2)g_{n,m}-2F^2_{n,m}g_{n,m}=H_+g_{n,m}. \quad (6)$$

The stability of a stationary solution $F_{n,m}$ is determined by the properties of the operators H_+ and H_-. Equations (5) and (6) can be rewritten in the form

$$-\frac{\partial^2 f_{n,m}}{\partial t^2}=H_+H_-f_{n,m}. \quad (7)$$

We will use the notations $(f,g)\equiv\Sigma_{n,m}f_{n,m}g_{n,m}$ and f without indices for $f_{n,m}$. The central point of this letter is the following instability theorem.

Let f be a solution of Eq. (7), where H_+ and H_- are self-adjoint (symmetric) operators with the following properties:

(1) $(f,H_+f)\geqslant 0$ and $(f,H_+f)=0$ only if $f=0$ or $f=F_{n,m}$;

(2) there exists some $\tilde{f}$ for which $(\tilde{f},H_-\tilde{f})<0$ under the constraint $(\tilde{f},F)=0$. Then there exists some f satisfying $(f,H_+f)\geqslant A\exp(2\gamma t)$ with constant A and

$$\gamma^2=\sup_{(f,F)=0}\frac{-(f,H_-f)}{(f,H_+^{-1}f)}. \quad (8)$$

A sketch of the proof is as follows. Using the properties of the operator H_+, Eq. (7) can be rewritten [for distribution f satisfying $(f,F)=0$] in the form

$$\frac{\partial^2 H_+^{-1} f_{n,m}}{\partial t^2} = -H_- f_{n,m}. \tag{9}$$

Multiplying Eq. (9) by $\partial_t f$ and summing over n and m, we obtain an integral of motion for the evolution equation (9):

$$(f_t, H_+^{-1} f_t) + (f, H_- f) = C = \text{const}. \tag{10}$$

Using assumption (2) we can consider solutions to Eq. (9) with $C=0$. Suppose that at $t=0$ the function f satisfies $(f_0, H_- f_0) < 0$, with the additional requirement $(f_0, F) = 0$. Let $f(t)|_{t=0} = f_0, f_t|_{t=0} = \tilde{\gamma} f_0$, where $\tilde{\gamma} > 0$ is defined by

$$\tilde{\gamma}^2 = -\frac{(f_0, H_- f_0)}{(f_0, H_+^{-1} f_0)}.$$

It is easy to check that $C=0$ for such a solution. Multiplying Eq. (9) by f, summing, and taking into account Eq. (10), after some straightforward algebra we obtain

$$\frac{\partial^2}{\partial t^2}(f, H_+^{-1} f) = 4(f_t, H_+^{-1} f_t) \geq \frac{4(f_t, H_+^{-1} f)^2}{(f, H_+^{-1} f)} = \frac{[\frac{\partial}{\partial t}(f, H_+^{-1} f)]^2}{(f, H_+^{-1} f)}. \tag{11}$$

Integration of this inequality yields

$$\frac{\frac{\partial}{\partial t}(f, H_+^{-1} f)}{(f, H_+^{-1} f)} \geq 2\tilde{\gamma} = \left.\frac{\frac{\partial}{\partial t}(f, H_+^{-1} f)}{(f, H_+^{-1} f)}\right|_{t=}. \tag{12}$$

Thus, $(f, H_+^{-1} f) \geq (f_0, H_+^{-1} f_0)\exp(2\tilde{\gamma} t)$, and the perturbation of the ground state $F_{n,m}$ grows in time. A maximum of the growth rate $\tilde{\gamma}$ of the instability is given by the γ defined in Eq. (8).

Obviously, the instability is triggered by the existence of a negative eigenvalue of the operator H_- under the additional constraint $(f, F) = 0$. Now we prove that in the DNLSE the operators H_- and H_+ satisfy all the requirements of the theorem, and instability of the ground state takes place when

$$\frac{\partial}{\partial \lambda^2}(F, F) = \frac{\partial}{\partial \lambda^2} P < 0.$$

Note that $H_+ F_{n,m} = 0$ and

$$H_- \frac{\partial F_{n,m}}{\partial \lambda^2} = -F_{n,m}.$$

First, we prove that the operator H_+ is non-negative. Indeed, H_+ can be written in a form

$$(H_+ f)_n = -f_{n+1,m} - f_{n-1,m} - f_{n,m+1} - f_{n,m-1} + \frac{F_{n,m+1} + F_{n,m-1}}{F_{n,m}} f_{n,m}$$
$$+ \frac{F_{n+1,m} + F_{n-1,m}}{F_{n,m}} f_{n,m}. \tag{13}$$

Thus,

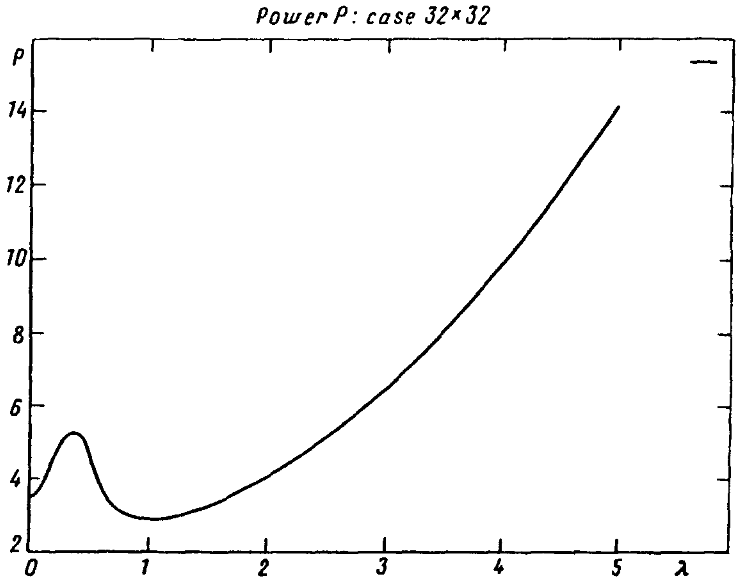

FIG. 1. Integral P versus λ for N=M=32. Solitons with $dP/d\lambda<0$ are unstable.

$$(f,H_+f)=\sum_{n,m}\left[\left(\sqrt{\frac{F_{n-1,m}}{F_{n,m}}}f_{n,m}-\sqrt{\frac{F_{n,m}}{F_{n-1,m}}}f_{n-1,m}\right)^2+\left(\sqrt{\frac{F_{n,m-1}}{F_{n,m}}}f_{n,m}-\sqrt{\frac{F_{n,m}}{F_{n,m-1}}}f_{n,m-1}\right)^2\right]\geqslant 0.$$

To demonstrate that there exists some $\tilde{f}$ satisfying $(\tilde{f},F)=0$ and $(\tilde{f},H_-\tilde{f})<0$, let us consider $s=\partial F/\partial\lambda^2-\alpha F$, where

$$\alpha=\frac{1}{2}\frac{d}{d\lambda^2}\ln(F,F).$$

The constraint $(s,F)=0$ is satisfied by construction of s. Using the obvious equalities

$$H_-\frac{\partial F_{n,m}}{\partial\lambda^2}=-F_{n,m}$$

and

$$H_-F=-4F^3,$$

we obtain

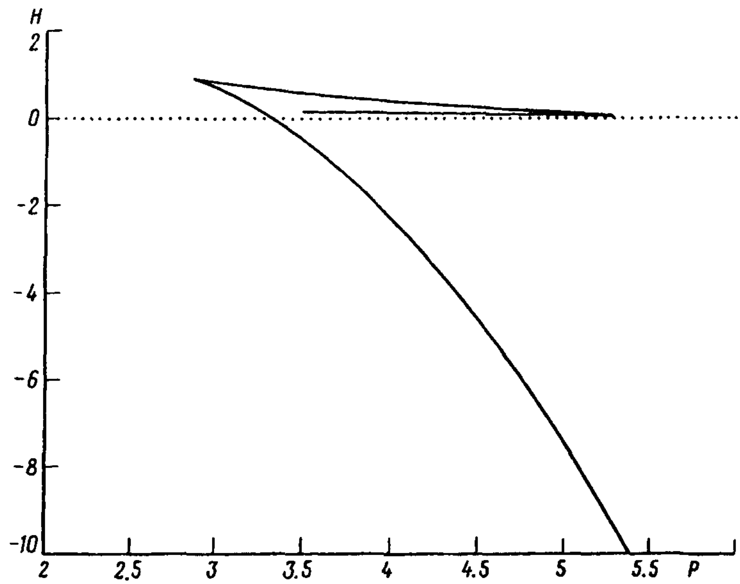

FIG. 2. Hamiltonian H as a function of P. The cuspidal edge corresponds to a degenerate critical point.

$$(s,H_{-}s) = -\left(F,\frac{\partial F}{\partial\lambda^2}\right) - 2\alpha\left(F,H_{-}\frac{\partial F}{\partial\lambda^2}\right) + \alpha^2(F,H_{-}F) = -(F,F^3)\left[\frac{d}{d\lambda^2}\ln(F,F)\right]^2 + \frac{d}{d\lambda^2}(F,F). \tag{14}$$

From this, one can see that a sufficient criterion for instability can be formulated as

$$\frac{d}{d\lambda^2}(F,F) = \frac{d}{d\lambda^2}P < 0.$$

Figure 1 shows the dependence of P on the spectral parameter λ. P is calculated for a finite number of modes. $P(\lambda)$ has been calculated[17] by a variational approach for infinite chains (the one-dimensional case with higher-order nonlinearity has been considered in Ref. 18). In this case, the left wing of the curve depicted in Fig. 1 degenerates into a point, and solitons are unstable for small λ. As was found previously,[15] an important new feature that is introduced by the discreteness is a coexistence of stable solitons and unstable states. The latter behavior is absent in the continuum limit of Eq. (1). We would like to point out that, in contrast to the continuum limit, moving solutions of Eq. (1) cannot be obtained by Gallilean transformation. Very narrow discrete states cannot move, but solitons close to the continuum limit can. Therefore, the stability of moving solitons is an open question, which we plan to consider in a forthcoming publication.

Our instability criterion agrees with predictions of the "catastrophe" theory. Figure 2 shows a plot of $H(P)$ numerically calculated soliton solutions. The occurrence of a

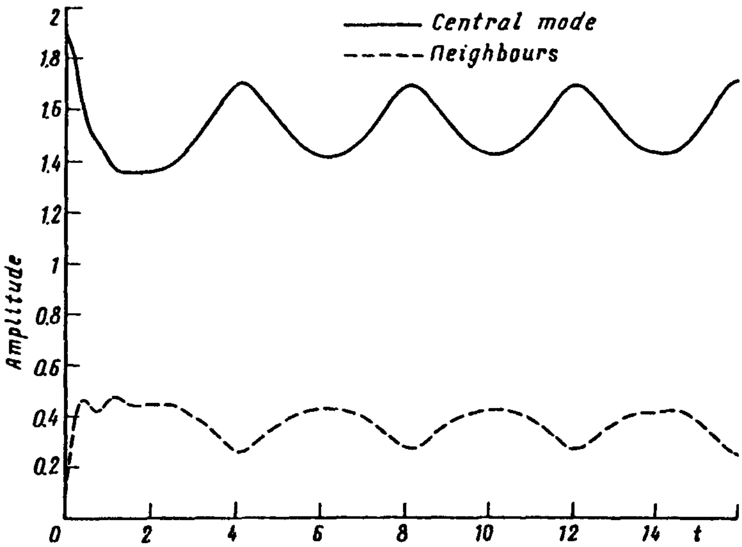

FIG. 3. Evolution of an initial distribution with all the energy in only one mode. The development into a narrow, breather-like solution can be described by Eqs. (15) and (16).

Whitney gather (Whitney surface) corresponds to the existence of unstable solitons with $(d/d\lambda^2)P<0$. They correspond to a saddle point of the Hamiltonian for fixed P. The cuspidal edge corresponds to a degenerate critical point (see, e.g., the review[19]).

The nonlinear development of an unstable discrete soliton leads to the formation of breather-like structures.[15] This type of the behavior is typical for rather general initial conditions.

One type of breather-like solutions can be described by a perturbation method, namely, very narrow breathing states. Let us assume that practically all the energy is localized in a few modes arranged in a symmetrical way with $\Psi_{0,0} \gg |\Psi_{0,\pm 1}|, |\Psi_{\pm 1,0}|$ We mark the central mode by $\Psi_0=\Psi_{0,0}$. Due to the symmetry, we can write also $\Psi_{\pm 1,0}=\Psi_{0,\pm 1}=\Psi_1$. The equation for central mode is approximately

$$i\frac{d\Psi_0}{dt}+4\Psi_1-4\Psi_0+2|\Psi_0|^2\Psi_0=0, \tag{15}$$

and for the nearest neighbors we obtain

$$i\frac{d\Psi_1}{dt}+\Psi_0-4\Psi_1+2|\Psi_1|^2\Psi_1=0. \tag{16}$$

In the second equation we neglect small corrections due the influence of other (small-amplitude) neighbors. There are two conserved quantities for Eqs. (15) and (16), namely $P=|\Psi_0|^2+4|\Psi_1|^2$ and $H=4|\Psi_0-\Psi_1|^2+12|\Psi_1|^2-|\Psi_0|^4-4|\Psi_1|^4$. This means that this system of equations is integrable. The exact solution can be expressed in terms of elliptic functions. Introducing $Z=|\Psi_0|^2-4|\Psi_1|^2$, we can easily rewrite Eqs. (15) and (16) in the form

$$Z_{tt}=-16Z-4\left(\frac{P+Z}{2}-\frac{P-Z}{8}\right)\left[H-4P+\left(\frac{P+Z}{2}\right)^2+4\left(\frac{P-Z}{8}\right)^2\right]=-\frac{\partial U(Z)}{\partial Z}. \tag{17}$$

The latter is equivalent to an equation for a particle moving in the potential $U(Z)$. This equation may be integrated once:

$$\frac{1}{2}Z_t^2+U(Z)=\frac{1}{2}Z_t^2+8Z^2+4(H-4P)\left[\left(\frac{P+Z}{2}\right)^2+4\left(\frac{P-Z}{8}\right)^2\right]+2\left[\left(\frac{P+Z}{2}\right)^2+4\left(\frac{P-Z}{8}\right)^2\right]^2=E. \tag{18}$$

Because we are mainly interested in the self-trapped states, consider as an initial condition a distribution for which at $t=0$ all energy is concentrated in the central mode, $Z=P, Z_t=0$, such that $H=4P-P^2$. In this case, the energy is $E=U(P)=8P^2-2P^4$. A solution can be found in terms of elliptic functions. A typical solution of Eq. (1) evolving into a narrow breather-like state is plotted in Fig. 3. We would like to note that the narrow states can be formed from initially broad distributions through either a quasi-collapse mechanism[15] or collisions of broad solitons.[20]

In conclusion, we have obtained an exact analytical criterion for the instability of discrete solitons in the two-dimensional discrete nonlinear Schrödinger equation. We have examined discrete, narrow, breather-like solutions.

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