The Supplementary Materials for the article: New approaches to coding information using inverse scattering transform

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NUMERICAL APPROACHES FOR INVERSE AND DIRECT SCATTERING TRANSFORM

In this paragraph we briefly overview and remind basic information concerning numerical Inverse Scattering Transform (IST) which is used in the main text of our paper [1]. First, we write down the Gelfand-Levitan-Marchenko equations (GLME) in the standard form for the left scattering problem at fixed distance (e.g. $z = 0$):

$$\begin{aligned} A_1^*(t,s) + \int_{-s}^{t} A_2(t,\tau)\Omega(s+\tau)d\tau = 0, \\ -A_2^*(t,s) + \int_{-s}^{t} A_1(t,\tau)\Omega(s+\tau)d\tau + \Omega(t+s) = 0, \\ \Omega(t) \equiv \Omega(z=0,t), \\ -t \leqslant s < t, \qquad 0 \leqslant t \leqslant T_s. \end{aligned} \quad (1)$$

Here $A_1(t,s)$ and $A_2(t,s)$ are the auxiliary complex functions that links together the kernel Ω and solution q of the NLSE via the GLME (1) and the following relation:

$$q(z=0,t) = -2A_2^*(t,t)\,. \quad (2)$$

The propagation problem is solved by the use of a simple formulae for scattering data evolution (see Eq. (5) and Eq. (11) in [1]).

In the general case numerical solution of an integral equation requires $\sim M^3$ operations (recall that M is the number of signal discretisation points). To reconstruct the whole signal $q(t_m)$ we need to perform this procedure at all points of the discrete grid (formula (7) in [1]) and, thus, the total cost $\sim M^4$ operations, that is not feasible for practical numerical implementation.

In this work we use the efficient Toeplitz inner-bordering (TIB) numerical scheme for both the inverse and direct scattering transform. Indeed, as it was shown Frumin and co-authors (reference [23] in [1]) the GLME (1) can be rewritten in the Toeplitz form by applying a simple transformation:

$$\begin{aligned} u(t,x) = A_1(t,t-x)\,, \\ v(t,y) = -A_2^*(t,y-t)\,. \end{aligned} \quad (3)$$

Now the GLME contains Toeplitz-type kernel $\Omega(y-x)$:

$$\begin{aligned} u(t,x) - \int_{-x}^{2t} \Omega^*(y-x)v(t,y)dy = 0, \\ v(t,y) + \int_{0}^{y} \Omega(y-x)u(t,x)dx + \Omega(y) = 0\,, \end{aligned} \quad (4)$$

and, as a result, the numerical TIB IST takes only M^2 operations (see details in reference [23] in [1]). Moreover, recently Frumin and co-authors have demonstrated (reference [24] in [1]) that the TIB algorithm can be reversely applied to the GLME (4), i.e. it allows to find the kernel $\Omega(t_m)$ from the known signal $q(t_m)$. Again, the required number of numerical operations is M^2. The numerical schemes and details can be found in references [23,24] in [1].

Here [1] we apply both inverse and direct TIB algorithm to the continuous spectrum signals. For the discrete spectrum case we apply only direct TIB method to recover the kernel, meanwhile to create signal at the beginning of the transmission line we use exact N-SS, described in the next paragraph.

N-soliton solutions of the NLSE

For the discrete spectrum kernel (see formula (3) in [1]) factorization of the GLME (1) leads to the system of linear algebraic equations (see, for instance the monograph of Lamb – reference [7] in [1]). Then, the N-SS can be found in the following exact form:

$$q^{(N)}(z=0,t) = -2\,\langle \mathbf{\Psi}(t)|\,(\widehat{\mathbf{E}} + \widehat{\mathbf{M}}(t)^*\widehat{\mathbf{M}}(t))^{-1}\,|\mathbf{\Phi}(t)\rangle\,. \quad (5)$$

Here $\widehat{\mathbf{E}}$ is $N \times N$ identity matrix,

$$\begin{aligned} \langle \mathbf{\Psi}(t)| = \left\langle c_1 e^{-i\xi_1 t}, ..., c_N e^{-i\xi_1 t}\right|\,, \\ \langle \mathbf{\Phi}(t)| = \left\langle e^{-i\xi_1 t}, ..., e^{-i\xi_1 t}\right|\,, \\ \widehat{\mathbf{M}}_{k,j}(t) = c_j \frac{e^{i(\xi_k^* - \xi_j)t}}{\xi_k^* - \xi_j}\,, \end{aligned} \quad (6)$$

and parameters c_k are defined in [1].

To the best of our knowledge all the existing discrete spectrum numerical IST algorithms are unstable at large N, that can be understood by looking at the exact N-SS formulae (5),(6). Indeed, the eigenvalues ξ_k are complex and, thus, the matrix $\widehat{\mathbf{E}} + \widehat{\mathbf{M}}(t)^*\widehat{\mathbf{M}}(t)$ in (5) may become ill-conditioned at large $|t|$. In such cases we use the arbitrary precision arithmetics to obtain accurate N-SS signal. Recently, A.A.Gelash and D.S. Agafontsev found that numerical realisation of the Zakharov-Shabat dressing method can be stably used up to $N \sim 32$ soliton solutions (see the reference [26] in [1]). Application of the dressing method to our kernel-based approach is a nontrivial task, however we believe that this can be an interesting direction for future research.

PARAMETRIC KERNEL DECODING

In this paragraph we discuss the N-SS kernel general parametric encoding/decoding schemes involving $4 \times N$ coding parameters. Let us write the N-SS kernel (formula (3) in [1]) as a time series on the discrete grid (see formula (7) in [1]):

$$\Omega_m \equiv \Omega(t_m) = \qquad (7)$$
$$= \sum_{k=1}^{N} c_k e^{-i\xi_k t_m} = \sum_{k=1}^{N} c_k e^{-i\xi_k T(m-1)} = \sum_{k=1}^{N} c_k z_k^{m-1}\,.$$

Parameters $z_k = \exp(-i\xi_k T)$ in (7) are defined by the soliton eigenvalues ξ_k and by the value of time slot T. Here, we again choose the minimum possible number of time samples $M = N$. Then, for the decoding problem we obtain system of equations with the Vandermonde matrix:

$$\begin{pmatrix} z_1^0, & z_2^0, & ..., & z_N^0 \\ z_1^1, & z_2^1, & ..., & z_N^1 \\ ... & ... & ... & ... \\ z_1^{N-1}, & z_2^{N-1}, & ..., & z_N^{N-1}, \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_N \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ ... \\ \Omega_N \end{pmatrix}. \qquad (8)$$

Now we consider both *position-phase* modulation and *amplitude-frequency* modulation of the N-SS kernel and discuss numerical problems that occur in general case.

Position-phase modulation

Suppose we know the eigenvalues ξ_k and hence the parameters $z_k = \exp(-i\xi_k)$. The decoding problem is to find the parameters c_k by the measured kernel samples Ω_i, that can be done by solving system (8). However, the straightforward numerical algorithm based, for example, on Gauss elimination in a general case is extremely challenging since the Vandermonde matrix (8) exponentially fast becomes ill-conditioned with the increase of N (see the reference [27] in [1]). On the other hand, the Vandermonde matrix belongs to the class of structured matrices for which the effective numerical algorithms have been developed (see the reference [28] in [1]). By applying the effective matrix inversion algorithm the kernel decoding can be performed using N^2 operations, however, the numerical stability restricts N by around ~ 60 harmonics (see, for example the reference [29] in [1]).

The inversion of the Vandermonde matrix becomes numerically stable at any N when z_k are the complex k-th roots of unity. For the N-SS kernel this is possible only when ξ_k have identical imaginary parts (that can be moved to the right part of the matrix system (8)), i.e. in the case presented by formula (6) in [1]. The additional harmonics orthogonality condition (formula (8) in [1]) allows us to use the FFT/IFFT algorithms instead of matrix inversion operations, that motivated us to focus on this elegant encoding scheme [1].

Amplitude-frequency modulation

Another possibility is to use the eigenvalues ξ_k as the carriers of information. They have to be found from the measured kernel samples Ω_m, while the shift-phase parameters c_k are all known and are not used for coding of information. The parametric approach based on the Prony's method (see, for example see the reference [30], chapter 11 in [1]) uses the following *master polynomial*

$$\phi(z) = \prod_{n=1}^{N} (z - z_k)^n = \sum_{n=0}^{N} a_n z^n, \quad a_0 = 1\,, \qquad (9)$$

with the complex roots z_k. The coefficients a_n of the polynomial (9) can be determined by solving the Toeplitz system of equations:

$$\begin{pmatrix} \Omega_N, & \Omega_{N-1}, & ..., & \Omega_1 \\ \Omega_{N+1}, & \Omega_N, & ..., & \Omega_2 \\ ... & ... & ... & ... \\ \Omega_{2N-1}, & \Omega_{2N-2}, & ..., & \Omega_N, \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_N \end{pmatrix} = \begin{pmatrix} \Omega_{N+1} \\ \Omega_{N+2} \\ ... \\ \Omega_{2N} \end{pmatrix}. \qquad (10)$$

Numerical solution of the problem (10) can be obtained by the use of Levinson-Durbin-Trench algorithm through the $O(N^2)$ arithmetic operations (see the reference [28] in [1]). However, the subsequent roots finding of the master polynomial $\phi(z)$ is the hard numerical problem for the large number of samples N. For example, the well known factorization algorithm of Jenkins and Traub becomes numerically unstable at $N \sim 100$ (see the reference [35] in [1]).

We note, that in the case of continuous spectrum kernel (formula (10) in [1]) the corresponding Vandermonde matrix can be always stably inverted since it becomes Fourier matrix.

We conclude that the general (parametric) N-SS kernel decoding requires matrix inversion and/or finding roots of the polynomial in the decoder. Although, the advanced numerical algorithms with a relatively small number of operations $\sim N^2$ can be exploited, their stability against large number of harmonics and additive noise requires a separate comprehensive analysis that is beyond the scope of this Letter.

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[1] New approaches to coding information using inverse scattering transform.